

THE DENSEST MATROIDS IN MINOR-CLOSED CLASSES WITH EXPONENTIAL GROWTH RATE

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ABSTRACT. The *growth rate function* for a nonempty minor-closed class of matroids \mathcal{M} is the function $h_{\mathcal{M}}(n)$ whose value at an integer $n \geq 0$ is defined to be the maximum number of elements in a simple matroid in \mathcal{M} of rank at most n . Geelen, Kabell, Kung and Whittle showed that, whenever $h_{\mathcal{M}}(2)$ is finite, the function $h_{\mathcal{M}}$ grows linearly, quadratically or exponentially in n (with base equal to a prime power q), up to a constant factor.

We prove that in the exponential case, there are nonnegative integers k and $d \leq \frac{q^{2k}-1}{q-1}$ such that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all sufficiently large n , and we characterise which matroids attain the growth rate function for large n . We also show that if \mathcal{M} is specified in a certain ‘natural’ way (by intersections of classes of matroids representable over different finite fields and/or by excluding a finite set of minors), then the constants k and d , as well as the point that ‘sufficiently large’ begins to apply to n , can be determined by a finite computation.

1. INTRODUCTION

We write $\varepsilon(M)$ for the number of points in a matroid M , so $\varepsilon(M) = |M|$ for every simple matroid M . For a nonempty minor-closed class of matroids \mathcal{M} , the *growth rate function* $h_{\mathcal{M}}(n): \mathbb{Z}_{\geq 0} \rightarrow \mathbb{Z} \cup \{\infty\}$ for \mathcal{M} is defined for all $n \geq 0$ by

$$h_{\mathcal{M}}(n) = \max\{\varepsilon(M): M \in \mathcal{M}, r(M) \leq n\}.$$

If \mathcal{M} contains all rank-2 uniform matroids, then clearly $h_{\mathcal{M}}(n) = \infty$ for all $n \geq 2$. Otherwise, $h_{\mathcal{M}}(n)$ is controlled up to a constant factor by the following theorem of Geelen, Kabell, Kung and Whittle [3]:

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Theorem 1.1 (Growth Rate Theorem). *Let \mathcal{M} be a minor-closed class of matroids not containing all rank-2 uniform matroids. There exists an integer α such that either*

- (1) $h_{\mathcal{M}}(n) \leq \alpha n$ for all $n \geq 0$,
- (2) $\binom{n+1}{2} \leq h_{\mathcal{M}}(n) \leq \alpha n^2$ for all $n \geq 0$ and \mathcal{M} contains all graphic matroids, or
- (3) there is a prime power q such that $\frac{q^n-1}{q-1} \leq h_{\mathcal{M}}(n) \leq \alpha q^n$ for all $n \geq 0$, and \mathcal{M} contains all $\text{GF}(q)$ -representable matroids.

Classes of type (3) are *(base- q) exponentially dense*. Our main result is an essentially best-possible refinement of condition (3), determining the precise value of each exponential growth rate function for all but finitely many n , and determining which large-rank matroids in \mathcal{M} attain this function.

Theorem 1.2. *Let \mathcal{M} be a base- q exponentially dense minor-closed class of matroids. There are integers $k \geq 0$ and $d \in \{0, 1, \dots, \frac{q^{2k}-1}{q^2-1}\}$ so that*

$$h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$$

for all sufficiently large n . Moreover, if $M \in \mathcal{M}$ has sufficiently large rank and satisfies $\varepsilon(M) = h_{\mathcal{M}}(r(M))$, then M is, up to simplification, a k -element projection of a projective geometry over $\text{GF}(q)$.

By this we mean that $\text{si}(M) \cong \text{si}(M')$ for some matroid M' obtained from a rank- $(r(M)+k)$ projective geometry over $\text{GF}(q)$ by k successive ‘extension-contraction’ operations. The theorem was conjectured in [4].

The ‘sufficiently large rank’ condition in Theorem 1.2 is necessary in general, as the union of a base- q exponentially dense minor-closed class \mathcal{M} with, say, the class of all $\text{GF}(q')$ -representable matroids of rank at most t for some fixed $q' > q$, is base- q exponentially dense, but has a growth rate function that only adopts base- q exponential behaviour for $n > t$. However, if \mathcal{M} is specified in some natural ‘finitary’ way, then one might expect to compute $h_{\mathcal{M}}(n)$ for every n . We prove a theorem showing that, in many cases, such a computation is possible in principle (as usual, $\text{Ex}(\mathcal{O})$ denotes the class of matroids with no minor isomorphic to a matroid in \mathcal{O}):

Theorem 1.3. *Let \mathcal{F} be a finite set of finite fields and \mathcal{O} be a finite set of simple matroids. Let \mathcal{M} be the class of matroids in $\text{Ex}(\mathcal{O})$ that are representable over all fields in \mathcal{F} . If \mathcal{M} is base- q exponentially dense and does not contain all truncations of $\text{GF}(q)$ -representable matroids, then there are computable nonnegative integers k, d and n_0 such that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq n_0$.*

Here, by *computable* we mean that there is a Turing machine which, given some encoding of \mathcal{F} and \mathcal{O} as input, will output k, d and n_0 in finite time. Since $h_{\mathcal{M}}(n)$ can also be computed by exhaustion for all $n < n_0$, the above theorem shows that the entire growth rate function can be computed precisely for any such class.

The insistence that the fields in \mathcal{F} are finite is artificial and just exists to avoid technicalities involving ‘specifying’ an infinite field; if \mathbb{F} -representability of a given matroid can be decided by a Turing machine for each $\mathbb{F} \in \mathcal{F}$, then the conclusion of Theorem 1.3 still holds. The ‘truncation’ condition, on the other hand, is necessary for our methods. Fortunately, this condition holds whenever $\mathcal{F} \neq \emptyset$ or \mathcal{O} contains a co-line (since the truncation of a large circuit is a large co-line, which is not representable over a small finite field), so Theorem 1.3 applies in both these natural cases.

If \mathcal{M} contains all truncations of $\text{GF}(q)$ representable matroids, then it appears that the densest matroids in \mathcal{M} , though they are just small projections of projective geometries, can be ‘wild’. We conjecture that the problems this causes for our proof methods are fundamental, and that Theorem 8.1 does not hold in full generality:

Conjecture 1.4. *Let \mathcal{O} be a finite set of matroids such that $\text{Ex}(\mathcal{O})$ is base- q exponentially dense for some prime power q . It is undecidable to determine whether there exists an integer $k \geq 0$ such that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1}$ for all sufficiently large n .*

2. PRELIMINARIES

We mostly use the notation of Oxley [9], but also write $|M|$ for $|E(M)|$ and $\varepsilon(M)$ for $|\text{si}(M)|$. Two matroids M, N are equal *up to simplification* if $\text{si}(M) \cong \text{si}(N)$. A *simplification* of M is a $\text{si}(M)$ -restriction of M (ie. any matroid obtained from M by deleting all loops and all but one element from each parallel class.)

An *elementary projection* (also called a *quotient*) of a matroid M is a matroid M' of the form \widehat{M}/e , where e is an element of a matroid \widehat{M} that is not a loop or coloop, such that $M = \widehat{M} \setminus e$. Thus, $r(M') = r(M) - 1$ and $E(M') = E(M)$. If $E(M)$ is the unique flat of M that spans e in \widehat{M} , then \widehat{M} is the *free extension* of M by e , and M' is the *truncation* of M ; we write $M' = T(M)$.

A *k-element projection* of M is a matroid obtained from M by a sequence of k elementary projections; it is easy to show that M' is a k -element projection of M if and only if there is a matroid \widehat{M} and a

k -element independent independent set K of \widehat{M} such that $M = \widehat{M} \setminus K$ and $M' = \widehat{M}/K$.

A collection \mathcal{X} of subsets in a matroid M is *skew* if $r_M(\cup_{X \in \mathcal{X}} X) = \sum_{X \in \mathcal{X}} r_M(X)$. A pair (X, Y) of sets in M is a *modular pair* in M if $r_M(X \cap Y) = r_M(X) + r_M(Y) - r_M(X \cup Y)$. A set X is *modular* in M if it forms a modular pair with every flat of M . For example, every flat in a projective geometry is modular. Modularity gives a sufficient condition for a certain type of ‘sum’ of two matroids to exist. If M and N are matroids, and the set $E(M) \cap E(N)$ is modular in M , then there is a unique matroid $M \oplus_m N$ with ground set $E(M) \cup E(N)$ that has both M and N as restrictions. We call this matroid, first defined by Brylawski [1], the *modular sum* of M and N , although many authors refer to the *generalised parallel connection*.

A *computable function* is one that can be calculated by a Turing machine that halts in finitely many steps; all functions defined in this paper are trivially computable. We also require that functions associated with the two theorems below, proved respectively in [2] and [5], are computable.

Theorem 2.1. *There is a computable function $f_{2.1}: \mathbb{Z}^3 \rightarrow \mathbb{Z}$ so that for every prime power q and all integers $\ell, m \geq 2$, if M is a matroid with no $U_{2,\ell+2}$ -minor satisfying $\varepsilon(M) \geq f_{2.1}(q, \ell, m)q^{r(M)}$, then M has a $\text{PG}(m-1, q')$ -minor for some $q' > q$.*

Theorem 2.2. *There is a computable function $f_{2.2}: \mathbb{Z}^3 \times \mathbb{Q} \rightarrow \mathbb{Z}$ so that for every prime power q , all $\beta \in \mathbb{Q}_{>0}$ and all integers $\ell, m \geq 2$, if M is a matroid with no $U_{2,\ell+2}$ -minor satisfying $r(M) \geq f_{2.2}(q, \ell, m, \beta)$ and $\varepsilon(M) \geq \beta q^{r(M)}$, then M has an $\text{AG}(m-1, q)$ -restriction or a $\text{PG}(m-1, q')$ -minor for some $q' > q$.*

Computability of $f_{2.1}$ follows from computability of the functions defined in ([2]: Lemmas 3.3, 4.3, 5.1 and 5.3). Computability of $f_{2.2}$ relies on computability of the functions in ([5]: Lemmas 4.3 and 5.2, Theorem 6.1) which themselves rely on computability of $f_{2.1}$, as well as that of the function defined in ([4], Lemma 8.1) and the main result of [7], the Density Hales-Jewett theorem. Checking the computability of all these functions directly is straightforward except for the theorem in [7], and in that theorem the authors take care to provide computable upper bounds for the associated function (see [7], Theorem 1.5).

3. GEOMETRIES AND PROJECTIONS

In this section we discuss the matroids that we show are the densest in classes in \mathbf{E}_q , namely the projections of projective geometries.

For each prime power q and every integer k , let $\mathcal{P}_q(k)$ denote the class of matroids of rank at least 2 that are, up to simplification, a loopless k -element projection of a projective geometry. $\mathcal{P}_q(0)$ is just the class of matroids that simplify to $\text{PG}(n, q)$ for some $n \geq 1$. In general, each rank- r matroid $M \in \mathcal{P}_q(k)$ satisfies $\text{si}(M) \cong \text{si}(\widehat{M}/K)$, where K is a rank- k independent **flat** of a rank- $(r+k)$ matroid \widehat{M} such that $\widehat{M} \setminus K \cong \text{PG}(r+k-1, q)$. (K is a flat because \widehat{M}/K is loopless.)

We will establish basic properties of matroids in $\mathcal{P}_q(k)$ regarding their density and local structure, then give a method of recognition.

3.1. Density. We now calculate the density of the matroids in $\mathcal{P}_q(k)$. To obtain a lower bound, we first show that a projective geometry cannot be nontrivially partitioned into a small number of flats:

Lemma 3.1. *If $G \cong \text{PG}(n-1, q)$ and \mathcal{F} is a partition of $E(G)$ into flats of G with $|\mathcal{F}| > 1$, then $|\mathcal{F}| \geq q^{n/2-1}$.*

Proof. We have $r_G(F) \leq n-1$ for all $n \in \mathcal{F}$, so $|G \setminus F| \geq |\text{AG}(n-1, q)| = q^{n-1}$ for each $F \in \mathcal{F}$. Since any two flats of G with rank greater than $n/2$ intersect, there is at most one such flat $F_0 \in \mathcal{F}$; if there is no such F_0 , let $F_0 \in \mathcal{F}$ be arbitrary. Now $\mathcal{F} - \{F_0\}$ is a partition of $E(G \setminus F_0)$ into flats of rank at most $n/2$. This gives

$$|\mathcal{F} - \{F_0\}| \geq \left(\frac{q^{n/2-1}-1}{q-1} \right)^{-1} |G \setminus F_0| > q^{-n/2} q^{n-1} = q^{n/2-1},$$

giving the result. \square

Lemma 3.2. *If M is a rank- r matroid in $\mathcal{P}_q(k)$, then*

- (1) *there exists $d \in \left\{0, \dots, \frac{q^{2k}-1}{q^2-1}\right\}$ such that $\varepsilon(M) = \frac{q^{r+k}-1}{q-1} - qd$,*
- (2) *M has a spanning restriction in $\mathcal{P}_q(k')$ for each $k' \leq k$, and*
- (3) *$\varepsilon(M) \geq q^{k/2}$.*

Proof. We may assume that M is (without simplification) a loopless k -element projection of $\text{PG}(r+k-1, q)$. Let \widehat{M} be a rank- $(r+k)$ matroid and K be a k -element independent flat of \widehat{M} such that $\widehat{M} \setminus K \cong \text{PG}(r+k-1, q)$ and $\widehat{M}/K = M$. Let $G = \widehat{M} \setminus K$.

If $k' \leq k$ and F is a rank- $(r+k')$ flat of G with $\cap_{\widehat{M}}(F, K) = k'$ (which is easily obtained by taking any rank- $(r+k')$ flat containing a rank- r flat that is skew to K), then K contains $(k-k')$ coloops of $\widehat{M}|(F \cup K)$ and it is easy to see that $M|F$ is a rank- r matroid in $\mathcal{P}_q(k')$, giving (2).

Note that every point of M is a parallel class. The points of M are a partition of $E(G)$ into $\varepsilon(M)$ flats of G , and $\varepsilon(M) > 1$, so $\varepsilon(M) \geq q^{(r+k)/2-1} \geq q^{k/2}$ by Lemma 3.1, giving (3). Let \mathcal{P} be the collection

of points of M containing more than one element of $E(M)$. We have $\varepsilon(G) - \varepsilon(M) = \sum_{P \in \mathcal{P}} (|P| - 1) \equiv 0 \pmod{q}$. For each $P \in \mathcal{P}$, let $d_P = r_{\widehat{M}}(P) \geq 2$ and let $d_{\max} = \max_{P \in \mathcal{P}} d_P$. Let F be the flat of M spanned by $\cup \mathcal{P}$, and let $\mathcal{P}_0 \subseteq \mathcal{P}$ be minimal so that $\cup \mathcal{P}_0$ spans F in M (note that $|\mathcal{P}_0| = r_M(F)$). We may choose \mathcal{P}_0 so that $d_P = d_{\max}$ for some $P \in \mathcal{P}_0$. Observe that $\varepsilon(M|F) \geq \left(\frac{q^{d_{\max}} - 1}{q - 1}\right)^{-1} \varepsilon(\widehat{M}|F)$. Now \mathcal{P}_0 is a skew set of points of M , every pair of flats of G is modular, and K is a flat of \widehat{M} ; it follows that \mathcal{P}_0 is a skew set of flats of G whose union spans F in G . Therefore $r_G(F) = \sum_{P \in \mathcal{P}_0} d_P$, so

$$r_M(F) = |\mathcal{P}_0| \leq \sum_{P \in \mathcal{P}_0} (d_P - 1) = r_G(F) - r_M(F) \leq k.$$

If $r_M(F) = k$, then equality holds above, so $r_G(F) = 2k$ and $d_P = 2$ for all $P \in \mathcal{P}_0$ and thus (by choice of \mathcal{P}_0) for all $P \in \mathcal{P}$. This gives $|F| = \frac{q^{2k} - 1}{q - 1}$ and $|\mathcal{P}| \leq (q + 1)^{-1}|F|$, so

$$0 \leq \varepsilon(G) - \varepsilon(M) \leq \frac{q}{q+1}|F| = q \frac{q^{2k} - 1}{q^2 - 1}.$$

If $r_M(F) < k$, then $r_G(F) < 2k$, and

$$0 \leq \varepsilon(G) - \varepsilon(M) \leq |F| \leq \frac{q^{2k-1} - 1}{q - 1} < q \frac{q^{2k} - 1}{q^2 - 1}.$$

Since $\varepsilon(G) = \frac{q^{r+k} - 1}{q - 1}$ and $\varepsilon(G) \equiv \varepsilon(M) \pmod{q}$, we have (1). \square

We now wish to show that, given a large matroid in $\mathcal{P}_q(k)$, all but a few single-element contractions give another matroid in $\mathcal{P}_q(k)$ with the same density. To do this, we first need a Ramsey-type lemma about large collections of flats in a projective geometry:

Lemma 3.3. *Let q be a prime power and $n \geq 1$, $t \geq 2$ and $s \geq 0$ be integers. If \mathcal{F} is a set of rank- s flats of $G \cong \text{PG}(n - 1, q)$ such that $|\mathcal{F}| \geq q^{ts^3}$, then there is a t -element set $\mathcal{F}_0 \subseteq \mathcal{F}$ and a flat F_0 of G such that $F_0 = \cap_{F \in \mathcal{F}_0} F$ and $\{F - F_0 : F \in \mathcal{F}_0\}$ is skew in G/F_0 .*

Proof. If $s = 0$, then the result holds vacuously since $|\mathcal{F}| \leq 1$. Suppose that $s > 0$ and the lemma holds for smaller s . Let \mathcal{F}' be a maximal skew subset of \mathcal{F} . If $|\mathcal{F}'| \geq s$ then the lemma holds, so we may assume that $|\mathcal{F}'| < t$. Let $H = \text{cl}_G(\cup \mathcal{F}')$. So $r_G(H) < ts$ and $|H| < q^{ts}$; note that the number of nonempty flats of $G|H$ of rank at most k is less than $\sum_{i=1}^s |H|^i < q^{ts(s+1)}$.

By maximality of \mathcal{F}' , each $F \in \mathcal{F}$ intersects H in such a flat, so there a nonempty flat H_0 of $G|H$ and a set $\mathcal{F}'' \subseteq \mathcal{F}$ such that $|\mathcal{F}''| \geq q^{-ts(s+1)}|\mathcal{F}|$ and $F \cap H = H_0$ for each $F \in \mathcal{F}''$. Let $j = r_G(H_0)$. Now $\{F - H_0 : F \in \mathcal{F}''\}$ is a collection of rank- $(s - j)$ flats in G/H_0 of size

$|\mathcal{F}''| \geq q^{ts^3 - ts(s+1)} > q^{t(s-j)^3}$. Since $\text{si}(G/H_0)$ is a projective geometry over $\text{GF}(q)$, the lemma holds by an inductive argument. \square

Now we argue that at most q^{58k^4} points alter the density of a matroid in $\mathcal{P}_q(k)$ when contracted:

Lemma 3.4. *Let q be a prime power and $k \geq 0$ be an integer. If $M \in \mathcal{P}_q(k)$, and d is the integer such that $\varepsilon(M) = \frac{q^{r(M)+k-1}}{q-1} - qd$, then there is a set $X \subseteq E(M)$ such that $\varepsilon(M|X) < q^{58k^4}$ and $\varepsilon(M/e) = \frac{q^{r(M/e)+k-1}}{q-1} - qd$ for all $e \in E(M) - X$.*

Proof. Let \widehat{M} be a matroid having a k -element independent set K such that $\widehat{M} \setminus K \cong \text{PG}(r(M) + k - 1, q)$, and $\text{si}(\widehat{M}/K) \cong \text{si}(M)$. Let $G = \widehat{M} \setminus K$. Since $M \in \mathcal{P}_q(k)$, the set K spans no element of G in \widehat{M} . We may assume that $k \geq 1$, since $\mathcal{P}_0(q)$ is the class of $\text{GF}(q)$ -representable matroids and both statements are easy in this case.

Let $\mathcal{F}_{-1} = \emptyset$, and for each $j \in \{0, 1, 2\}$, let \mathcal{F}_j be the set comprising every flat F of G so that $\cap_{\widehat{M}}(F, K) = r_G(F) - j$ and $\cap_{\widehat{M}}(F', K) \leq r_G(F') - j$ for each flat F' contained in F . Since K spans no element of G , we have $\mathcal{F}_0 = \emptyset$. The flats in \mathcal{F}_1 correspond exactly to the points of \widehat{M}/K containing more than one element of \widehat{M} , so we have $qd = \frac{q^{r(M)+k-1}}{q-1} - \varepsilon(M) = \sum_{F \in \mathcal{F}_1} (|F| - 1)$. For each $e \in E(\widehat{M})$, if e lies in no flat in \mathcal{F}_1 or \mathcal{F}_2 , then the flats of rank at least 2 in G/e that are contracted onto points of $(\widehat{M}/K)/e$ correspond exactly to the flats in \mathcal{F}_1 (contracting e creates no new flat of this type, and destroys no flat of this type), so we have $qd = \frac{q^{r(M/e)+k-1}}{q-1} - \varepsilon(M/e)$. Let $X = \cup(\mathcal{F}_1 \cup \mathcal{F}_2)$; it suffices to show that $\varepsilon(M|X) < q^{58k^4}$.

Clearly every flat in $\mathcal{F}_1 \cup \mathcal{F}_2$ has rank at least 2 and at most $k+1$. If $|\mathcal{F}_1 \cup \mathcal{F}_2| \geq 2kq^{(k+1)(k+2)^3}$, then by Lemma 3.3 there is some $j \in \{1, 2\}$, some $t \in \{2, \dots, k+1\}$, and a $(k+1)$ -element set $\mathcal{H} \subseteq \mathcal{F}_j$ of rank- t flats of G such that $\{H - H_0 : H \in \mathcal{H}\}$ is a skew set in G/H_0 , where $H_0 = \cap \mathcal{H}$. Note that $r_G(H_0) \leq s-1$. If $j=1$ then, since K spans no element of H_0 we have $\cap_G(K, H_0) \leq s-2$. If $j=2$ then, since $H_0 \subseteq H \in \mathcal{F}_2$ for each $H \in \mathcal{H}_0$, we have $\cap_G(K, H_0) \leq r_G(H_0) - 2 = s-3$. In either case, $\cap_G(K, H_0) < \cap_G(K, H)$ for each $H \in \mathcal{H}$. It follows that $\{H - H_0 : H \in \mathcal{H}\}$ is a skew set of $k+1$ flats of G/H_0 , none of which is skew to K in G/H_0 . Since $r_{G/H_0}(K) \leq k$, this is a contradiction. Therefore $|\mathcal{F}_1 \cup \mathcal{F}_2| \leq 2kq^{(k+1)(k+2)^3}$. Since every flat in $\mathcal{F}_1 \cup \mathcal{F}_2$ has rank at most $k+1$, we have

$$\varepsilon(M|X) \leq \varepsilon(G|X) \leq 2kq^{k+1}q^{(k+1)(k+2)^3} < q^{4k^4}q^{(2k)(3k)^3} = q^{58k^4},$$

as required. \square

3.2. Local Representability. There are many different k -element projections of a projective geometry over $\text{GF}(q)$; some contain small restrictions that are not $\text{GF}(q)$ -representable (for example, the principal truncation of a plane of $\text{PG}(n, q)$ is in $\mathcal{P}_q(1)$ and has a U_{2, q^2+q+1} -restriction) and some do not (for example, the truncation of $\text{PG}(n, q)$ is in $\mathcal{P}_q(1)$ but contains no non- $\text{GF}(q)$ -representable flat of rank less than n). We define a parameter measuring the degree of ‘local representability’ of a matroid in $\mathcal{P}_q(k)$. For an integer $h \geq 1$, let $\mathcal{P}_q(k, h)$ be the class of matroids in $\mathcal{P}_q(k)$ having the property that every restriction of rank at most h is $\text{GF}(q)$ -representable.

This property can be easily described in terms of the projection itself:

Lemma 3.5. *Let $h \geq 2$ be an integer, and let $M \in \mathcal{P}_q(k)$. Let $\text{si}(M) \cong \text{si}(\widehat{M}/K)$ for some rank- $(r+k)$ matroid \widehat{M} and k -element independent set K of \widehat{M} so that $G = \widehat{M} \setminus K \cong \text{PG}(r+k-1, q)$. Then $M \in \mathcal{P}_q(k, h)$ if and only if $\square_{\widehat{M}}(K, F) = 0$ for every flat F of G of rank at most $h+1$.*

Proof. Suppose that $\square_{\widehat{M}}(K, F) > 0$ for some flat F of G with $r_G(W) \leq h+1$. Choose F to be minimal with this property, so $\square_{\widehat{M}}(K, F) = 1 < r_G(F)$. Let $F' = F$ if $r_G(F) > 2$, and if $r_G(F) = 2$, let F' be a plane of G containing F and so that $\square_{\widehat{M}}(F', K) = 1$; such a plane exists because $r(M) > 1$ so $r_{\widehat{M}}(K) \leq r+k-2$. By modularity of the flats in G and the fact that K spans no point of F' , there is at most one line of $G|F'$ that is not skew to K in G ; it follows that

$$\varepsilon(M|F') \geq \varepsilon(G|F') - q \geq \frac{q^{r_G(F')-1}-1}{q-1} - q > \frac{q^{r_G(F')-1}-1}{q-1} \geq \frac{q^{r_M(F')-1}-1}{q-1},$$

so $M|F'$ is too dense to be $\text{GF}(q)$ -representable. We have $r_M(F') \leq \max(h, 3-1) \leq h$, so $M \notin \mathcal{P}_q(k)$.

Conversely, suppose that $\square_{\widehat{M}}(K, F) = 0$ for every flat F of rank at most $h+1$ in G . For every independent set I of M with $r_M(I) \leq h$, we have $\square_{\widehat{M}}(I \cup \{e\}, K) = 0$ for all $e \in E(G)$, and so $e \notin \text{cl}_{\widehat{M}/K}(I)$ for all $e \in E(G) - \text{cl}_G(I)$. Therefore $M| \text{cl}_M(I) = G| \text{cl}_G(I)$ and thus $M| \text{cl}_M(I) \cong \text{PG}(r_M(I)-1, q)$, giving $M \in \mathcal{P}_q(k, h)$. \square

In particular, it follows from the above lemma that if $M \in \mathcal{P}_q(k, 2)$, then $\varepsilon(M) = \frac{q^{r(M)+k-1}-1}{q-1}$, since K is skew to every line of G and its contraction thus identifies no pair of points.

We now show that given a set X of at least half the elements a very large matroid in $\mathcal{P}_q(k, h)$, we can find a large contraction-minor in $\mathcal{P}_q(k, h)$ that is ‘covered’ by X .

Lemma 3.6. *There is a computable function $f_{3.6}: \mathbb{Z}^4 \rightarrow \mathbb{Z}$ so that, for every prime power q and all integers $k, m \geq 0$ and $h \geq 2$, if $M \in \mathcal{P}_q(k, h)$ is simple with $r(M) \geq f_{3.6}(q, k, h, m)$, and $X \subseteq E(M)$ satisfies $|X| \geq \frac{1}{2}|M|$, then there is a set $C \subseteq E(M)$ such that $M/C \in \mathcal{P}_q(k, h)$, $r(M/C) \geq m$, and each parallel class of M/C intersects X .*

Proof. Let q be a prime power and $k, h, m \geq 0$ be integers. Let $m_0 = \max(m, (k+1)(h+2)^3 + h+2)$. Set $f_{3.6}(q, k, h, m) = f_{2.2}(q, q, (2q)^{-1}, m_0)$.

Let $M \in \mathcal{P}_q(k, h)$ be simple with $r = r(M) \geq f_{3.6}(q, k, h, m)$, and let $X \subseteq E(M)$ satisfy $|X| \geq \frac{1}{2}|M|$. Let \widehat{M} be a matroid such that $G = \widehat{M} \setminus K \cong \text{PG}(r+k-1, q)$ and $\widehat{M}/K = M$ for some k -element independent set K of M that is skew in M to every flat of rank at most $h+1$ in G . We have $|X| \geq \frac{1}{2}|M| \geq \frac{1}{2}q^{r(\widehat{M})-1}$. Since G has no $U_{2,q+2}$ -minor, Theorem 2.2 implies that $G|X$ has an $\text{AG}(m_0, q)$ -restriction R .

Let $C_0 \subseteq E(G)$ be maximal so that

- (i) R is a restriction of G/C_0 , and
- (ii) K is skew to in \widehat{M}/C_0 to every set of rank at most $h+1$ in G/C_0 .

Let J be the set of elements $e \in E(G/C_0)$ such that $C_0 \cup \{e\}$ does not satisfy (ii).

3.6.1. $\varepsilon(M/C_0|J) < q^{m_0}$.

Proof of claim: Suppose otherwise. Let $G' \cong \text{PG}(r(G')-1, q)$ be a simplification of G/C_0 , and let $J' = J \cap E(G')$. For each $e \in J'$ there is some rank- $(h+1)$ flat H_e of G'/e such that $\cap_{G'/e}(H_e, K) > 0$, and so $F_e = H_e \cup \{e\}$ is a rank- $(h+2)$ flat of G' with $\cap_{G'}(F_e, K) > 0$. Let $\mathcal{F} = \{F_e : e \in J'\}$.

Since each flat in \mathcal{F} has size less than q^{h+2} , we have $|\mathcal{F}| > q^{-h-2}|J'| \geq q^{m_0-h-2} \geq q^{(k+1)(h+2)^3}$. By Lemma 3.3, there is a $(k+1)$ -element set $\mathcal{F}' \subseteq \mathcal{F}$ and a flat F_0 of G' such that $F_0 = \cap \mathcal{F}$ and the flats $\{F - F_0 : f \in \mathcal{F}'\}$ are skew in G'/F_0 . Since $r_N(F_0) \leq h$ we have $\cap_{G'}(F_0, K) = 0$, so $\{F - F_0 : f \in \mathcal{F}'\}$ is a skew collection of $k+1$ sets in G' , none of which is skew to K in G'/F_0 . This contradicts the fact that $r_{G'/F_0}(K) = r_{G'}(K) = k$. \square

If R is not spanning in \widehat{M}/C_0 , then $E(R)$ spans a proper subflat of the projective geometry G/C_0 , so there are at least $q^{m_0+1} > \varepsilon(\widehat{M}/C_0|J)$ points of G/C_0 not spanned by $E(R)$; contracting any such point not in J gives a contradiction to the maximality of C . Therefore R is spanning in \widehat{M}/C_0 . Similarly, $|R| = q^{m_0} > \varepsilon((\widehat{M}/C_0)|J)$, so there is some $e \in$

$E(R) - J$; let $C = C_0 \cup \{e\}$. Now $E(R)$ contains a simplification R' of G/C , and K is a rank- k flat of \widehat{M}/C skew to every flat of R' of rank at most $h + 1$; it follows that $M/C = (\widehat{M}/(C \cup K)) \in \mathcal{P}_q(k, h)$. Since $E(R') \subseteq X$ contains a simplification of G/C , the lemma follows. \square

3.3. Recognition. We now prove a result that will identify matroids in $\mathcal{P}_q(k)$. We use the fact (see [9], Proposition 7.3.6) that, if M and N are matroids on a common ground set E , then N is a projection of M if and only if $\text{cl}_M(X) \subseteq \text{cl}_N(X)$ for all $X \subseteq E$. Since we are concerned with isomorphism and simplification as well, it is convenient to give a slightly different version of this statement. If M and N are matroids then we say that $\varphi: E(M) \rightarrow E(N)$ is a *projective map* from M to N if $\varphi(\text{cl}_M(X)) \subseteq \text{cl}_N(\varphi(X))$ for all $X \subseteq E(M)$. It is routine to show that, if φ is a projective map from M to N , then the matroid $N|_{\varphi(E(M))}$ is isomorphic to the simplification of a projection of M ; the set of elements of M that map to a given element corresponds to a parallel class in this projection of M .

We use this to identify projections of projective geometries, as well as projections of representable matroids that are the union of at most two flats of a projective geometry. If $G \cong \text{PG}(n-1, q)$ in the following lemma, then the conclusion gives $M \in \mathcal{P}_q(k)$ for some k .

Lemma 3.7. *Let M be a simple matroid and let G be a restriction of $\text{PG}(n-1, q)$ such that $E(G)$ is the union of at most two flats of $\text{PG}(n-1, q)$. If there is a surjective map $\varphi: E(G) \rightarrow E(M)$ such that, for every triangle T of G , either $|\varphi(T)| = 1$ or $\varphi(T)$ is a triangle of M , then M is, up to simplification, a projection of G .*

Proof. It suffices to show that $\varphi(\text{cl}_G(X)) \subseteq \text{cl}_M(\varphi(X))$ for all $X \subseteq E(G)$. Let $X \subseteq E(G)$ and let F_1, F_2 be flats of $\text{PG}(n-1, q)$ so that $E(G) = F_1 \cup F_2$. Let $X_i = X \cap F_i$. It is easy to see that every element of $\text{cl}_G(X_1 \cup X_2)$ is either in $\text{cl}_G(X_1) \cup \text{cl}_G(X_2)$, or lies in a triangle of G containing a point of $\text{cl}_G(X_1)$ and a point of $\text{cl}_G(X_2)$. It follows routinely that there is a sequence of sets

$$X_1 \cup X_2 = Y_0, Y_1, Y_2, \dots, Y_m = \text{cl}_G(X_1 \cup X_2)$$

so that, for each $i \in \{1, \dots, m\}$, we have $Y_i = Y_{i-1} \cup \{e_i\}$ for some e_i that is contained in a triangle of $G|_{Y_i}$. An easy inductive argument implies that $\varphi(Y_i) \subseteq \text{cl}_M(\varphi(X_1 \cup X_2))$ for each i . Therefore $\varphi(\text{cl}_G(X_1 \cup X_2)) \subseteq \text{cl}_M(\varphi(X_1 \cup X_2))$, as required. \square

We now prove the lemma allowing us to recognise general matroids in $\mathcal{P}_q(k)$ from a modularity assumption.

Lemma 3.8. *Let q be a prime power, and $\ell \geq 2$ and $j, t \geq 0$ be integers. Let $s = 10\ell + t$. If M is a matroid of rank at least $2s + t$ with no $U_{2,\ell}$ -restriction, and K is a rank- t subset of $E(M)$ such that $M/K \in \mathcal{P}_q(j, s)$, and for every rank- $(t+1)$ flat of M containing K and every line L of M , the pair (L, K) is modular, then there exists $D \subseteq \text{cl}_M(K)$ such that $M \setminus D \in \mathcal{P}_q(k)$ for some k .*

Proof. We may assume that M is simple and that K is a flat of M .

3.8.1. *Every flat F of M with $r_M(F) \leq 2s$ and $\cap_M(F, K) = 0$ satisfies $M|F \cong \text{PG}(r_M(F) - 1, q)$.*

Proof of claim: Let $r = r_M(F) \leq 2s$. Now $M|F$ is a rank- r restriction of $(M/K)|\text{cl}_{M/K}(F)$, which is a rank- r flat of a matroid in $\mathcal{P}_q(j, s)$ so is isomorphic to $\text{PG}(r - 1, q)$. Moreover, for all distinct $e, f \in F$, the line $\text{cl}_M(\{e, f\})$ intersects $\text{cl}_M(K \cup \{x\})$ for all $x \in \text{cl}_{M/K}(\{e, f\})$, as otherwise this pair of flats fails to be modular. Therefore $M|F$ is a simple rank- r restriction of $\text{PG}(r - 1, q)$ in which every line contains at least $q + 1$ points; it follows that $M|F \cong \text{PG}(r - 1, q)$. \square

Let F_0 be a rank- $2s$ flat of M that is skew to K , and let $X_0 \subseteq E(M)$ be maximal such that $F_0 \subseteq X_0$ and $M|X_0 \in \mathcal{P}_q(k)$ for some k . Let $G_0 \cong \text{PG}(n - 1, q)$, and let $\varphi_0: E(G_0) \rightarrow X_0$ be the surjective map associated with this projection. Let $G \cong \text{PG}(n, q)$ be an extension of G_0 . If $E(M) = X_0 \cup K$, then the lemma clearly holds, so we may assume that there is some $e \in E(M) - (X_0 \cup K)$.

3.8.2. *There is a rank- s flat \widehat{X} of M , containing e and skew to K , so that $M|(\widehat{X} \cap X_0) \cong \text{PG}(s - 2, q)$ and $M|\widehat{X} \cong \text{PG}(s - 1, q)$.*

Proof of claim: Note that $r_M(X_0) \geq 2s \geq s + t$; let Y be a rank- $(s - 1)$ flat of $M|X_0$ that is skew to K in M/e . Since $M|X_0 \in \mathcal{P}_q(k)$, the matroid $M|Y$ has a $\text{PG}(s - 2, q)$ -restriction. However, $M|Y$ is contained in a $\text{PG}(s - 2, q)$ -restriction of M by the first claim, so $M|Y \cong \text{PG}(s - 2, q)$ and Y is a flat of M . Let $\widehat{X} = \text{cl}_M(Y \cup \{e\})$. Now \widehat{X} is skew to K in M , and by the first claim, $M|\widehat{X} \cong \text{PG}(s - 1, q)$, and Y is a hyperplane of $M|\widehat{X}$. If $f \in (\widehat{X} - Y) \cap X_0$, then $\text{cl}_M(\{e, f\})$ meets Y in some f_0 , so $e \in \text{cl}_M(\{f, f_0\}) - X_0$. Since $\text{cl}_M(\{f, f_0\})$ is a line of X_0 , it contains $q + 1$ points of X_0 , so contains at least $q + 2$ points of M . Since $\text{cl}_M(\{f, f_0\})$ is skew to K , this contradicts the first claim. Therefore $Y = \widehat{X} \cap X_0$, and the claim follows. \square

Let \widehat{G} be a $\text{PG}(s - 1, q)$ -restriction of G so that $G|(E(\widehat{G}) \cap E(G_0)) \cong \text{PG}(s - 2, q)$. Let $\widehat{\varphi}: E(\widehat{G}) \rightarrow \widehat{X}$ be a matroid isomorphism between

\widehat{G} and $M|_{\widehat{X}}$ so that $\widehat{\varphi}(x) = \varphi(x)$ for all $x \in E(\widehat{G}) \cap E(G_0)$. Let $a = \widehat{\varphi}^{-1}(e)$, noting that $a \notin E(G_0)$.

3.8.3. *For each $b \in E(G) - \{a\}$, there is some $x(b) \in E(M)$ and a 10-element independent set $J(b)$ of $\widehat{G} \setminus a$ such that*

- (i) $x(b) \in \text{cl}_M(\{e, \varphi_0(b_0)\})$ where $\{b_0\} = \text{cl}_G(\{a, b_0\}) \cap E(G_0)$, and $x(b) = \varphi_0(b_0)$ if and only if $b = b_0$, and
- (ii) for all $c \in J(b)$, there is a triangle $\{c, d_0, b\}$ of G such that $d_0 \in E(G_0)$ and $\{\widehat{\varphi}(c), \varphi_0(d_0), x(b)\}$ is a triangle of M , and
- (iii) if $b \in E(\widehat{G})$ then $x(b) = \widehat{\varphi}(b)$, and if $b \in E(G_0)$ then $x(b) = \varphi_0(b)$.

Proof of claim: If $b \in E(G_0)$ then set $x(b) = \varphi_0(b)$ and set $J(b)$ to be an arbitrary 10-element independent set of $G_0 \setminus b$; the conditions follow easily for $x(b)$ and $J(b)$ since $b \notin J(b)$ and φ_0 is a projective map. If $b \in E(\widehat{G}) - E(G_0)$, then set $x(b) = \widehat{\varphi}(b)$ and $J(b)$ to be an arbitrary 10-element independent set of $\widehat{G} \setminus (\text{cl}_G(\{a, b\}) \cup (E(G_0) \cap E(\widehat{G})))$; again, the conditions easily follow from the fact that $\widehat{\varphi}$ is an isomorphism and $E(G_0) \cap E(\widehat{G})$ is a hyperplane of \widehat{G} .

Suppose that $b \in E(G) - (E(G_0) \cup E(\widehat{G}))$. Let $\{b_0\} = \text{cl}_G(\{a, b\}) \cup E(G_0)$. Note that $r(\widehat{G} \setminus (E(G_0) \cap E(\widehat{G}))) = s \geq 10\ell + t \geq 9\ell + t + 3$; let J be a $(9\ell + t + 2)$ -element independent set of \widehat{G}/a that is disjoint to $E(G_0)$. we will choose $J(b)$ to be an appropriate subset of J . Let $c \in J$, and let $\{c_0\} = \text{cl}_G(\{a, c\}) \cap E(G_0)$. The set $\{c_0 : c \in J\}$ is a $|J|$ -element independent set of \widehat{G} , so there is at most one $c \in J$ for which $\varphi_0(c_0) = \varphi_0(b_0)$; let $J' \subseteq J$ be a $(9\ell + t + 1)$ -element set containing no such c . Let $c \in J'$, and $\{d_0\} = \text{cl}_G(\{c, b\}) \cap E(G_0)$. Since $\widehat{\varphi}$ is an isomorphism, the set $\widehat{\varphi}(\{c_0, c, a\})$ is a triangle of M . Since φ_0 is projective and $b_0 \neq c_0$, the set $\varphi_0(\{c_0, d_0, b_0\})$ is also a triangle of M . Now $e \notin \text{cl}_M(\varphi_0(G_0))$, so the latter triangle does not span e in M ; hence, $\text{cl}_M(\{\widehat{\varphi}(c), \varphi_0(d_0)\})$ and $\text{cl}_M(\{e, \varphi_0(b_0)\})$ are distinct coplanar lines of M .

If $\{e, \varphi_0(b_0)\}$ is not skew to K in M , then it is contained in a rank- $(t + 1)$ flat of M containing K , and it follows from the hypothesis that for each $c \in J'$, the coplanar lines $\text{cl}_M(\{e, \varphi_0(b_0)\})$ and $\text{cl}_M(\{\widehat{\varphi}(c), \varphi_0(d_0)\})$ form a modular pair, and so intersect at some $x_c \in E(M)$. If $\{e, \varphi_0(b_0)\}$ is skew to K in M , then since $M|_{\widehat{\varphi}(J')} \cong \widehat{G}|_{J'}$, the set $\widehat{\varphi}(J')$ is independent in M , so there is a $(9\ell + 1)$ -element subset J'' of J' such that the plane $\text{cl}_M(\{e, \varphi_0(b_0), \widehat{\varphi}(c)\})$ is skew to K in M for each $c \in J''$. By the first claim, the restriction of M to such a plane is isomorphic to $\text{PG}(2, q)$, and so again the lines $\text{cl}_M(\{\widehat{\varphi}(c), \varphi_0(d_0)\})$ and $\text{cl}_M(\{e, \varphi_0(b_0)\})$ intersect at a point $x_c \in E(M)$.

In either case above, there is a $(9\ell + 1)$ -element subset J' of J such that the point $x_c \in \text{cl}_M(\{e, \varphi_0(c_0)\})$ is well-defined for all $c \in J'$. This line contains at most ℓ elements of M , so there is a 10-element set $J(b) \subseteq J''$ and some $x \in E(M)$ so that $x_c = x$ for all $c \in J(b)$. Since $\{e, \widehat{\varphi}(c), \varphi_0(c_0)\}$ and $\{\varphi_0(c_0), \varphi_0(d_0), \varphi_0(b_0)\}$ are non-collinear triangles of M and $\{\widehat{\varphi}(c), \varphi_0(d_0), x(b)\}$ is a triangle of M , we have $x(b) \neq \varphi_0(b_0)$. Now (i),(ii) and (iii) follow from construction. \square

Define $\varphi: E(G) \rightarrow E(M)$ by $\varphi(a) = \{e\}$ and $\varphi(b) = x(b)$ for each $b \in E(G) - a$, noting that φ is an extension of both $\widehat{\varphi}$ and φ_0 .

3.8.4. *For each triangle T of G , either $|\varphi(T)| = 1$ or $\varphi(T)$ is a triangle of M .*

Proof of claim: This is clear if $T \subseteq E(\widehat{G}) \cup E(G_0)$, since any such T is a triangle of either \widehat{G} or G_0 , and both $\widehat{\varphi}$ and φ_0 are projective maps. It follows from this observation and Lemma 3.7 that $\varphi|(E(G_0) \cup E(\widehat{G}))$ is a projective map from $G|(E(G_0) \cup E(\widehat{G}))$ to M .

Let $T = \{b^1, b^2, b^3\}$ be a triangle of G . If $b^1 = a$, then let $\{b_0\} = E(G_0) \cap \text{cl}_G(T)$. Note that $\varphi(b_0) \in X_0$ and $\varphi(a) = e \notin X_0$, and that b^2 and b^3 are not both b_0 ; it follows from 3.8.3(i) that $\varphi(T)$ is a triangle of G . We may thus assume that $a \notin T$.

Let $i \in \{1, 2, 3\}$. We have $M|\varphi(J(b^i)) \cong \widehat{G}|J(b^i)$, so $\varphi(J(b^i))$ is a 10-element independent set of M . We can therefore choose elements $c^i: i \in \{1, 2, 3\}$ so that $c^i \in J(b^i)$ for each i , and so that

- $c^i \notin \text{cl}_G(T \cup \{c_1, \dots, c_{i-1}\})$ for each $i \in \{1, 2, 3\}$, and
- $\varphi(c^i) \notin \text{cl}_M(\varphi(T) \cup \{\varphi(c^1), \dots, \varphi(c^{i-1})\})$ for each $i \in \{1, 2, 3\}$.

(This choice is possible since for each i the set of invalid c^i has rank at most 4 in G and the set of invalid $\varphi(c^i)$ has rank at most 5 in M .)

For $i \in \{1, 2, 3\}$, let $x^i = \varphi(b^i)$, let $y^i = \varphi(c^i)$, and let $T^i = \{c^i, d_0^i, b^i\}$ be a triangle of G so that $d_0^i \in E(G_0)$ and $\varphi(T^i)$ is a triangle of M . Let $z^i = \varphi(d_0^i)$ for each i , and let $W = \{c^1, c^2, c^3, d_0^1, d_0^2, d_0^3\}$. By construction of the c^i , we have $G|W \cong U_{5,6}$.

If $\varphi(T)$ is not a triangle or singleton of M , then either $M|\varphi(T) \cong U_{3,3}$ or $M|\varphi(T) \cong U_{2,2}$. In the former case, we have $r_M(\varphi(T) \cup \{y^1, y^2, y^3\}) = 6$ so, since $\{x^i, y^i, z^i\}$ is a triangle of M for each i , we also have $M|\varphi(W) \cong U_{6,6}$. If $M|\varphi(T) \cong U_{2,2}$, then we may assume that $x^1 = x^2 \neq x^3$. By choice of y^1, y^2, y^3 , we therefore have $M|\varphi(W) \cong U_{3,4} \oplus U_{2,2}$, where $\{y^1, y^2, z^1, z^2\}$ is the four-element circuit. Neither $U_{6,6}$ nor $U_{3,4} \oplus U_{2,2}$ is projection of $M|W \cong U_{5,6}$; since $W \subseteq E(G_0) \cup E(\widehat{G})$, we thus have a contradiction to the fact that $\varphi|E(G_0) \cup E(\widehat{G})$ is a projective map from $G|(E(G_0) \cup E(\widehat{G}))$ to M . \square

It follows from Lemma 3.7 that $M|\varphi(E(G)) \in \mathcal{P}_q(k)$ for some k ; since $X_0 \cup \{e\} \subseteq \varphi(E(G))$, this contradicts the maximality of X_0 . \square

4. EXPONENTIALLY DENSE CLASSES

Let \mathbf{E}_q denote the collection of all base- q exponentially dense minor-closed classes of matroids. Each class in \mathbf{E}_q , by definition, does not contain arbitrarily long lines or arbitrarily large projective geometries over fields larger than $\text{GF}(q)$. Such classes are also ‘controlled’ in other ways that we describe in the following lemma, stating them all in terms of a single parameter c for convenience. We will refer freely to the parameter $c = c_{\mathcal{M}}$ for a given $\mathcal{M} \in \mathbf{E}_q$ throughout the paper.

Lemma 4.1. *For every $\mathcal{M} \in \mathbf{E}_q$ there is an integer $c = c_{\mathcal{M}}$ so that*

- (1) $U_{2,c+2} \notin \mathcal{M}$.
- (2) $\text{PG}(c-1, q') \notin \mathcal{M}$ for all $q' > q$,
- (3) $\varepsilon(M) < q^{r(M)+c}$ for all $M \in \mathcal{M}$,
- (4) $\varepsilon(M/C) > q^{-c-r_M(C)}\varepsilon(M)$ for all $M \in \mathcal{M}$ and $C \subseteq E(M)$, and
- (5) $\mathcal{M} \cap \mathcal{P}_q(k) = \emptyset$ for all $k \geq c$.

Moreover, there is a computable function $f_{4.1}: \mathbb{Z}^3 \rightarrow \mathbb{Z}$ so that, if $\ell \geq 2$ and $s \geq 2$ are integers such that $U_{2,\ell+2} \notin \mathcal{M}$ and $\text{PG}(s-1, q') \notin \mathcal{M}$ for all $q' > q$, then $c_{\mathcal{M}} \leq f_{4.1}(q, \ell, s)$.

Proof. Since $\mathcal{M} \in \mathbf{E}_q$, there are integers $\ell, s \geq 2$ such that $U_{2,\ell+2} \notin \mathcal{M}$ and $\text{PG}(s-1, q') \notin \mathcal{M}$ for all $q' > q$. (This is true because $U_{2,q'+1} \cong \text{PG}(1, q') \notin \mathcal{M}$ for all $q' > \ell$, and \mathcal{M} does not contain all $\text{GF}(q')$ -representable matroids for any $q' \in \{q+1, \dots, \ell\}$). We show that c can be defined to depend computably on q and these two parameters; indeed, let $d = \lceil \log_q(f_{2.1}(q, \ell, c_2)) \rceil$ and set $c = \max(\ell, s, q^{2d+4})$.

Clearly c satisfies (1) and (2) and, by Theorem 2.1, we have $\varepsilon(M) \leq q^{r(M)+d} < q^{r(M)+c}$ for all $M \in \mathcal{M}$, so c satisfies (3). Let $C \subseteq E(M)$ and \mathcal{F} be the collection of rank- $(r_M(C)+1)$ flats of M containing C ; we have $\varepsilon(M) < \sum_{F \in \mathcal{F}} \varepsilon(M|F) \leq \varepsilon(M/C)q^{r_M(C)+d+1} < q^{r_M(C)+c}\varepsilon(M/C)$, so c satisfies (4).

Suppose that $k \geq q^{2d+4}$ and $M \in \mathcal{M} \cap \mathcal{P}_q(k)$. By Lemma 3.2 we have $q^{r(M)+d} > \varepsilon(M) \geq q^{2d+4}$, so $r(M) \geq d+4$. Now M has a spanning restriction in $\mathcal{P}_q(d+2)$, giving

$$\varepsilon(M) \geq \frac{q^{r(M)+d+2}-1}{q-1} - q \frac{q^{2(d+2)}-1}{q^2-1} > q^{r(M)+d+1} - q^{2d+4} > q^{r(M)+d},$$

a contradiction. Since $c \geq q^{2d+4}$, it follows that c satisfies (5). \square

4.1. Connectivity. A rank- r matroid is *weakly round* if every cocircuit has rank at least $r - 1$. This is a very strong connectivity property that is a slight relaxation of *roundness*, a notion introduced by Kung (where cocircuits are required to be spanning) under the name of *non-splitting* in [6]. Our first lemma shows that an exponentially dense matroid of large rank in a class in \mathbf{E}_q has a comparably dense restriction of large rank that is also weakly round:

Lemma 4.2. *There is a computable function $f_{4.2}: \mathbb{Z}^4 \rightarrow \mathbb{Z}$ so that, for every prime power q and all integers $c, b, m \geq 0$, if $\mathcal{M} \in \mathbf{E}_q$ satisfies $c_{\mathcal{M}} \leq c$ and $g: \mathbb{Z} \rightarrow \mathbb{Z}$ is a function so that $g(n) > qg(n-1) \geq q^{b-1}$ for all $n > b$, then every $M \in \mathcal{M}$ satisfying $r(M) \geq f_{4.2}(q, c, b, m)$ and $\varepsilon(M) \geq g(r(M))$ has a weakly round restriction M' of rank at least m , such that either $M' = M$ or $\varepsilon(M') > g(r(M'))$.*

Proof. Let q be a prime power, and $b, m, c \geq 0$ be integers. Let $\varphi = \frac{1}{2}(1 + \sqrt{5})$. Let t be an integer so that $(q\varphi^{-1})^t > q^{c+1}$, and set $f_{4.2}(q, c, b, m) = n = \max(b, m + t)$.

Let $\mathcal{M} \in \mathbf{E}_q$ satisfy $c > c_{\mathcal{M}}$. Let $M \in \mathcal{M}$ satisfy $r = r(M) \geq n$ and $\varepsilon(M) \geq g(r)$. Let M' be a minimal restriction of M such that $E(M')$ is a flat of M and $\varepsilon(M') \geq \varphi^{r(M')-r} \varepsilon(M)$. Let $r' = r(M')$. If M' is not weakly round, then there are flats F_1 and F_2 of M' with $F_1 \cup F_2 = E(M')$ and $r(M'|F_i) \leq r' - i$ for each $i \in \{1, 2\}$; by minimality this gives $\varepsilon(M') \leq \varepsilon(M'|F_1) + \varepsilon(M'|F_2) < (\varphi^{-1} + \varphi^{-2})\varphi^{r'-r} \varepsilon(M) \leq \varepsilon(M')$, a contradiction. So M' is weakly round.

Note that, since $r \geq b$, we have $g(r) \geq q^{r-1}$, so

$$q^{r'+c} \geq \varepsilon(M') \geq (q\varphi^{-1})^{r-r'} q^{r'-r} \varepsilon(M) \geq (q\varphi^{-1})^{r-r'} q^{r'-1}.$$

Therefore $q^{-1}(q\varphi^{-1})^{r-r'} \leq q^c$, giving $r - r' \leq t$ and so $r' \geq n - t \geq m$.

If $r' = r$, then $M' = M$ since $E(M')$ is a flat of M ; since $r(M) \geq n \geq m$ the lemma holds. If $r' < r$, then $\varepsilon(M') \geq \varphi^{r'-r} g(r) = (q\varphi^{-1})^{r-r'} q^{r'-r} g(r) > g(r')$, since $r - r' > 0$ and $g(r) > q^{r-r'} g(r')$ by definition of g ; again, the lemma holds. \square

Weak roundness is clearly closed under contraction, and is thus easy to exploit to contract two restrictions of different ranks together:

Lemma 4.3. *If M is a weakly round matroid and X and Y are subsets of $E(M)$ such that $r_M(X) < r_M(Y)$, then M has a minor N so that $M|X = N|X$, $M|Y = N|Y$ and Y is spanning in N .*

Proof. Let N be a minimal minor of M so that $M|X = N|X$, $M|Y = N|Y$ and N is weakly round. By minimality we have $\varepsilon(N) = \text{cl}_N(X) \cup \text{cl}_N(Y)$. If Y is not spanning in N then $r_N(Y) \leq r(N) - 1$ and $r_N(X) \leq r(N) - 2$; these facts together contradict weak roundness of N . \square

4.2. Stacks. We will use Lemma 4.3 to contract two ‘incompatible’ restrictions together in a large matroid: an affine geometry over $\text{GF}(q)$, and a stack. For each prime power q and integers $k \geq 0$ and $t \geq 2$, a matroid S is a (q, k, t) -stack if there are disjoint sets $F_1, \dots, F_k \subseteq E(S)$ such that the union of the F_i is spanning in S and, for each $i \in \{1, \dots, k\}$ the matroid $(S/(F_1 \cup \dots \cup F_{i-1}))|_{F_i}$ has rank at most t and is not $\text{GF}(q)$ -representable.

Note that a (q, k, t) -stack has rank at least $2k$ and at most tk . Stacks are far from being $\text{GF}(q)$ -representable; our first lemma, proved in ([5], Lemma 4.2), shows that a stack of height $\binom{k+1}{2}$ on top of a large projective geometry guarantees a rank- k flat disjoint from the geometry.

Lemma 4.4. *Let q be a prime power and h and t be nonnegative integers. If M is a matroid with a $\text{PG}(r(M) - 1, q)$ -restriction R and a $(q, \binom{k+1}{2}, t)$ -stack restriction S , then $E(S) - E(R)$ contains a rank- k flat of M .*

Such a flat gives rise to a k -element projection of R . Since matroids in a given class $\mathcal{M} \in \mathbf{E}_q$ do not contain these projections for arbitrarily large k , the above implies that a large stack and a large affine geometry restriction cannot coexist in a weakly round matroid in \mathcal{M} :

Lemma 4.5. *Let $\mathcal{M} \in \mathbf{E}_q$, let $c = c_{\mathcal{M}}$, and let $s \geq 2$ be an integer. Then \mathcal{M} contains no weakly round matroid with an $\text{AG}(c^2s, q)$ -restriction and a (q, c^2, s) -stack restriction.*

Proof. Let $c = c_{\mathcal{M}}$ and let $M \in \mathcal{M}$ be weakly round with an $\text{AG}(c^2s, q)$ -restriction R and a (q, c^2, s) -stack restriction S . Since $r(S) \leq c^2s = r(R) - 1$, Lemma 4.3 gives a minor N of M with S as a restriction and R as a spanning restriction. Since $r(S) < r(R)$ there is some $e \in E(R) - \text{cl}_N(E(S))$; now N/e has S as a restriction and $(N/e)|(E(R/e))$ has a $\text{PG}(r(N/e) - 1, q)$ -restriction R' . Since $c^2 \geq \binom{c+1}{2}$, the matroid S has a $(q, \binom{c+1}{2}, s)$ -stack restriction, so there is a rank- c flat F of M disjoint from $E(R')$ by Lemma 4.4; now $r(R') > c + 1$; it follows that $(M/F)|_{E(R')} \in \mathcal{M} \cap \mathcal{P}_q(c)$, contradicting Lemma 4.1. \square

The affine geometries in the lemma above will be obtained from Theorem 2.2. The following is a more convenient version of the theorem that applies within a particular class in \mathbf{E}_q ; the equivalence, with $f_{4.6}(q, c, m, \beta) = f_{2.2}(q, c, \max(m, c + 1), \beta)$, is easy to see.

Theorem 4.6. *There is a computable function $f_{4.6}: \mathbb{Z}^3 \times \mathbb{Q} \rightarrow \mathbb{Z}$ so that, for every prime power q , all integers $c, m \geq 2$ and all $\beta \in \mathbb{Q}_{>0}$, if $\mathcal{M} \in \mathbf{E}_q$ satisfies $c_{\mathcal{M}} \leq c$, and $M \in \mathcal{M}$ satisfies $r(M) \geq f_{4.6}(q, c, m, \beta)$ and $\varepsilon(M) \geq \beta q^{r(M)}$, then M has an $\text{AG}(m - 1, q)$ -restriction.*

5. MINIMALITY

The results in the previous section imply that a dense matroid in a class $\mathcal{M} \in \mathbf{E}_q$ has a dense weakly round restriction that itself has a large affine geometry restriction. The following lemma, roughly, shows that a matroid which is minor-minimal with respect to being dense and having this geometry as a restriction, has a spanning projective geometry after contracting just a few elements.

Lemma 5.1. *There is a computable function $f_{5.1}: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ so that for every prime power q and integer $c \geq 2$, if $\mathcal{M} \in \mathbf{E}_q$ satisfies $c_{\mathcal{M}} \leq c$, and $M \in \mathcal{M}$ is weakly round, satisfies $\varepsilon(M) \geq q^{r(M)-1}$, and has an $\text{AG}(f_{5.1}(q, c) - 1, q)$ -restriction R such that $\varepsilon(M) > q\varepsilon(M/e)$ for all $e \in E(M) - \text{cl}_M(E(R))$, then there is a set $C \subseteq E(M)$ such that M/C has a $\text{PG}(r(M/C) - 1, q)$ -restriction and $r_M(C) \leq f_{5.1}(q, c)$.*

Proof. Let q be a prime power and $c \geq 2$ be an integer. Let $s = q^{5c^2} + 4$. Let

$$\beta = q^{-1}, \quad \beta_1 = \frac{1}{2}\beta q^{-c^2s-c}, \quad \beta_2 = \frac{1}{2}\beta_1 q^{-2q^{5c^2}}.$$

Let n_0 and t be integers so that $\beta_1 q^{n_0} \geq q^{c^2s+c}$ and $\beta_1 q^{n_0-c^2s-q^{5c^2}} \geq 2$, and $\beta_2 > q^{c-t}$. Finally, set

$$f_{5.1}(q, c) = \max(n_0, c^2s + 1, 20(c + 2) + c^2s + q^{4c^2} + 5).$$

Let $\mathcal{M} \in \mathbf{E}_q$ satisfy $c_{\mathcal{M}} \leq c$, let $n \geq f_{5.1}(q, c)$ be an integer, and let $M \in \mathcal{M}$ be weakly round such that $\varepsilon(M) \geq q^{r(M)-1}$ and M has an $\text{AG}(n - 1, q)$ -restriction R so that $\varepsilon(M) > q\varepsilon(M/e)$ for all $e \in E(M) - \text{cl}_M(E(R))$. We may assume that M is simple. If $r(M) < n + n_0$ then we can contract at most $n_0 - 1$ elements so that R is a spanning restriction and contract a further element of R to obtain a spanning projective geometry; since $n_0 \leq f_{5.1}(q, c)$ the lemma is satisfied. We may therefore assume that $r(M) \geq n + n_0$.

We say a line of a matroid is *long* if it contains at least $q + 2$ points, and *short* if it contains at most q points. Let \mathcal{L}_0 be a maximal skew collection of long lines of M and let $F = \text{cl}_M(\cup \mathcal{L}_0)$. Note that $M|F$ is a $(|\mathcal{L}_0|, q, 2)$ -stack and is therefore also a $(|\mathcal{L}_0|, q, s)$ -stack; by Lemma 4.5, M has no (c^2, q, s) -stack restriction, so $|\mathcal{L}_0| < c^2$ and $r_M(F) = 2|\mathcal{L}_0| < 2c^2$. By maximality of \mathcal{L}_0 , no long line of M is skew to F . For each $e \in E(M) - F$ it follows that the number of long lines of M through e does not exceed $\varepsilon((M/e)|\text{cl}_{M/e}(F)) < q^{2c^2+c} \leq q^{3c^2}$. We claim that every point not spanned by F or $E(R)$ is on few short lines:

5.1.1. *Each $e \in E(M) - (F \cup \text{cl}_M(E(R)))$ is in fewer than q^{4c^2} short lines of M .*

Proof of claim: For each such e , let \mathcal{L}^- and \mathcal{L}^+ respectively denote the sets of short and long lines of M through e , and let $\mathcal{L}^=$ be the set of all other lines through e . Note that $|\mathcal{L}^+| \leq q^{3c^2}$ and $\varepsilon(M/e) = |\mathcal{L}^+| + |\mathcal{L}^-| + |\mathcal{L}^=|$. Now

$$\begin{aligned} \varepsilon(M) &= \sum_{L \in \mathcal{L}^-} (|L| - 1) + q|\mathcal{L}^=| + \sum_{L \in \mathcal{L}^+} (|L| - 1) \\ &\leq (q - 1)|\mathcal{L}^-| + q|\mathcal{L}^=| + (c + 1)|\mathcal{L}^+| \\ &= q\varepsilon(M/e) + (c + 1 - q)|\mathcal{L}^+| - |\mathcal{L}^-| \end{aligned}$$

Now $\varepsilon(M) > q\varepsilon(M/e)$, so $|\mathcal{L}^-| < (c + 1 - q)|\mathcal{L}^+| < cq^{3c^2} < q^{4c^2}$. \square

Recall that $M|F$ is a $(|\mathcal{L}_0|, q, s)$ -stack. Let S be an (j, q, s) -stack restriction of M containing F for which j is as large as possible. By Lemma 4.5 we have $j < c^2$, so $r_M(S) < c^2s$. Let $M_1 = M/E(S)$, noting that $r(M_1) > r(M) - c^2s$; by maximality of j we know that every rank- s restriction of M_1 is $\text{GF}(q)$ -representable. We also have $\varepsilon(M_1) \geq q^{-c^2s-c}\varepsilon(M) \geq q^{-c^2s-c}\beta q^{r(M)} \geq 2\beta_1 q^{r(M_1)}$.

Let $X = \text{cl}_{M_1}(E(M_1) \cap E(R))$; note that $\varepsilon(M_1|X) \leq q^{n+c} \leq \beta_1 q^{n+n_0-c^2s} \leq \beta_1 q^{r(M)-c^2s} \leq \beta_1 q^{r(M_1)}$, so $\varepsilon(M_1 \setminus X) \geq \beta_1 q^{r(M_1)}$. Moreover, every nonloop of $M_1 \setminus X$ is in fewer than q^{4c^2} q -short lines of M and is therefore in fewer than q^{4c^2} short lines of M_1 .

5.1.2. *There are disjoint sets $J, Z \subseteq E(M_1)$, satisfying $r_{M_1}(J) \leq q^{4c^2}$ and $\varepsilon((M_1/J)|Z) \geq \beta_2 q^{r(M_1/J)}$, so that every short line of M_1/J is disjoint to Z .*

Proof of claim: For each nonloop e of M_1 , let J_e be the closure in M_1 of the union of the short lines of M_1 containing e . If $e \notin X$ then there are fewer than q^{4c^2} such lines, so $r(M_1|J_e) \leq q^{4c^2}$ and $\varepsilon(M_1|J_e) < q^{q^{4c^2}+c} < q^{q^{5c^2}}$. Recall that $\varepsilon(M_1 \setminus X) \geq \beta q^{r(M_1)}$; let Z' be a set of nonloops of $M_1 \setminus X$ satisfying $|Z'| = \lfloor \beta_1 q^{-q^{5c^2}} q^{r(M_1)} \rfloor$. We have $\sum_{z \in Z'} \varepsilon(M_1|J_z) < \beta_1 q^{r(M_1)} \leq \varepsilon(M_1 \setminus X)$; let y be a nonloop of $M_1 \setminus X$ such that $y \notin J_z$ for all $z \in Z'$.

Let $J = J_y$. We now argue that no nonloop of $(M_1/J)|(Z' - J)$ is in a short line. Suppose otherwise; let $\text{cl}_{M_1/J}(\{e, f\})$ be a short line of M_1/J with $e \in Z'$. The line $\text{cl}_{M_1}(\{e, f\})$ is also short in M_1 . Since $f \notin J$ we know that the line $\text{cl}_{M_1}(\{y, f\})$ is not short, so there is some triangle $\{y, f', f\}$ of M . If $\text{cl}_{M_1}(\{e, f'\})$ is short then $\text{cl}_{M_1}(\{e, f\})$ and $\text{cl}_{M_1}(\{e, f'\})$ are two short lines of M_1 whose union spans y ; this contradicts $y \notin J_e$. Therefore $\text{cl}_{M_1}(\{e, f'\})$ is not short in M_1 . But $y \in J$ so f' and f are parallel in M_1/Y ; it follows that $\text{cl}_{M_1}(\{e, f\})$ is not short in M_1/J , contradicting its definition.

Therefore no short line of M_1/J contains a nonloop of $(M_1/J)|Z'$. However, using $r(M_1|J) \leq q^{4c^2}$ and Lemma 4.1, we have

$$\begin{aligned} \varepsilon((M_1/J)|(Z' - J)) &\geq q^{-q^{4c^2} - c} \varepsilon(M_1|Z') \\ &\geq q^{-q^{5c^2}} \lfloor \beta_1 q^{-q^{5c^2}} q^{r(M_1)} \rfloor \\ &\geq \frac{1}{2} \beta_1 q^{-2q^{5c^2}} q^{r(M_1/J)}, \end{aligned}$$

where we use $r(M_1) \geq r(M_1/J)$ and the fact that $\beta_1 q^{-q^{5c^2}} q^{r(M_1)} \geq \beta_1 q^{n_0 - c^2 s - q^{5c^2}} \geq 2$. Thus, J and $Z = Z' - J$ satisfy the claim. \square

Let $M_2 = M_1/(J \cup \{f\})$ for some $f \in E(R)$. Since every rank- s restriction of M_1 is $\text{GF}(q)$ -representable and $r_{M_1}(J) \leq s - 4$, every rank-3 restriction of M_2 is $\text{GF}(q)$ -representable; in particular, M_2 has no long lines. We now build a nearly spanning projective geometry restriction in M_2 using Z .

5.1.3. M_2 has a $\text{PG}(r(M_2) - t - 1, q)$ -restriction.

Proof. Note that $(M/f)|E(R)$ has a projective geometry restriction of rank at least $20(q + 2) + c^2 s + q^{4c^2} + 2$. Since M_2 is obtained from M/f by contracting a set of rank at most $c^2 s + q^{4c^2}$, we see that M_2 has a $\text{PG}(20(q + 2) + 2, q)$ -restriction. Let m be maximal so that M_2 has a $\text{PG}(m - 1, q)$ -restriction G , so $m \geq 20(q + 2) + 2$; assume that $m < r(M_2) - t$. We have $\varepsilon(M_2|\text{cl}_{M_2}(G)) \leq q^{r(G)+c} < q^{c-t} q^{r(M_2)} \leq \beta_2 q^{r(M_2)}$, so there is some $e \in Z - \text{cl}_{M_1}(E(G))$. Let $G' = M|(\cup_{x \in E(G)} \text{cl}_{M_2}(\{e, x\}))$.

We will now apply Lemma 3.8 to G' . We have $\text{si}(G'/e) \cong G \in \mathcal{P}_q(0) = \mathcal{P}_q(0, 10(q + 2) + 1)$, and $r(G') = m + 1 \geq 20(q + 2) + 3$. By choice of e , every line of G' through e contains exactly $q + 1$ elements, and if there is a line of G' through e that is not modular to some other line of G' , then we can contract a point of the latter line to find a $U_{2, q+2}$ -restriction; this contradicts the fact that every rank-3 restriction of M_2 is $\text{GF}(q)$ -representable. Therefore (L, L') is a modular pair for every pair of lines of G' with $e \in L$. Lemma 3.8 applied with $\ell = q + 2$ and $t = 1$ now gives $G' \setminus D \in \mathcal{P}_q(k)$ for some $k \geq 0$ and $D \subseteq \text{cl}_{G'}(\{e\})$. Since matroids in $\mathcal{P}_q(k)$ have no coloop, it follows that G' has a spanning restriction in $\mathcal{P}_q(k)$, and so has a $\text{PG}(r(G') - 1, q)$ -restriction. This contradicts the maximality of m . \square

Let G be such a restriction and B be a basis of M_2 containing a basis B_G for G . Let $C = E(S) \cup J \cup (B - B_G)$. We have $M/C = M_2/(B - B_G)$,

so G is a $\text{PG}(r(M/C) - 1, q)$ -restriction of M/C . Moreover

$$r_M(C) \leq r_M(S) + r_{M_1}(J) + |B_G| \leq c^2 s + (s - 3) + t \leq f_{5.1}(q, c),$$

as required. \square

6. FINDING A PROJECTION

In this section, we show that a very high-rank matroid M_0 that is a few contractions away from being a very highly locally representable matroid in $\mathcal{P}_q(j)$ for some j is either in $\mathcal{P}_q(k)$ for some k , or contains a high-rank minor in $\mathcal{P}_q(k)$ that is, in a certain sense, denser than M_0 .

Lemma 6.1. *There is a computable function $f_{6.1}: \mathbb{Z}^5 \rightarrow \mathbb{Z}$ so that, for every prime power q and all integers $c, t, j, m \geq 0$, if $\mathcal{M} \in \mathbf{E}_q$ satisfies $c_{\mathcal{M}} \leq c$, and $M_0 \in \mathcal{M}$ satisfies $r(M_0) \geq f_{6.1}(q, c, t, j, m)$ and $M_0/K \in \mathcal{P}_q(j, f_{6.1}(q, c, t, j, m))$ for some rank- t set K of M_0 , then there is an integer $k \geq 0$ and a minor M of M_0 of rank at least m , so that $M \in \mathcal{P}_q(k)$ and either $M = M_0$, or*

$$\frac{q^{r(M)+k}-1}{q-1} - \varepsilon(M) < \frac{q^{r(M_0)+k}-1}{q-1} - \varepsilon(M_0).$$

Proof. Let $c, t, j, m \geq 0$ be integers. Let $\delta = q^{-c-t-4}$. Let $b = \delta^{-1} q^{t+1+c} = q^{2t+2c+5}$, and let \mathbf{I} be the interval $[-b, b] \subseteq \mathbb{Z}$.

Let $u = 10(c+2)+t$. Let $h: \mathbf{I}^4 \rightarrow \mathbb{Z}$ be a function so that $h(\mathbf{i}) \geq u$ for all $\mathbf{i} \in \mathbf{I}^4$ and h is strictly decreasing with respect to the lexicographic order on \mathbf{I}^4 . Let $n_0 \geq \max(m+1, 2c+3t-j+5, 2u+t)$ be an integer so that $q^{n_0+k} > q^{2c} + q^{c+t}$,

$$\delta \frac{q^{r+t-j}-1}{q-1} > q^{t+c+1} \quad \text{and} \quad |2(q^{k+t-j} - 1 - (q-1)s)| < q^{r-t+j} - 1$$

for all $r \geq n_0$, all $0 \leq k < c$ and all s with $|s| \leq q^{t+c+1} + q^{2c}$.

Let $n: \mathbf{I}^4 \rightarrow \mathbb{Z}$ be a function that is strictly decreasing with respect to the lexicographic order on \mathbf{I}^4 , and additionally satisfies $n(\mathbf{i}) \geq n_0$ for all $\mathbf{i} \in \mathbf{I}^4$, and

$$n(i_1, i_2, i_3, i_4) \geq t + f_{3.6}(q, j, h(i_1, i_2, i_3, i_4), n(i_1, i_2, i_3, i_4 + 1))$$

whenever $(i_1, i_2, i_3, i_4), (i_1, i_2, i_3, i_4 + 1) \in \mathbf{I}^4$. Set $f_{6.1}(q, c, t, j, m) = \max_{\mathbf{i} \in \mathbf{I}^4} (\max(h(\mathbf{i}), n(\mathbf{i})))$.

Let $\mathcal{M} \in \mathbf{E}_q$ be such that $c_{\mathcal{M}} \leq c$. Let $M_0 \in \mathcal{M}$ be a matroid with $r_0 = r(M_0) \geq f_{6.1}(q, c, t, j, m)$ and $M_0/K \in \mathcal{P}_q(k, f_{6.1}(q, c, t, j, m))$. For each minor N of M_0 for which $r_N(K) = t$ and $N/K \in \mathcal{P}_q(k, h')$ for some $h' \geq 2$, we define a 4-tuple $\sigma(N)$ measuring the ‘symmetry’ and ‘modularity’ of N relative to the set K . For such a minor N , let G_N be a simplification of N/K , let $A_{N,K}(x) = \text{cl}_N(\{x\} \cup K) - \text{cl}_N(K)$, and let $a_{N,K}(x) = \varepsilon(N|A_{N,K}(x))$. The sets $A_{N,K}(x): x \notin \text{cl}_N(K)$ correspond

to elements of G_N ; let $\mu(N), \nu(N)$ and $\rho(N)$ respectively denote the mean, minimum, and maximum of $a_{N,K}(x)$ over all $x \in E(G_N)$. Let

$$\begin{aligned}\sigma_1(N) &= \lfloor \delta^{-1}(\mu(N) - \mu(M_0)) \rfloor, \\ \sigma_2(N) &= \varepsilon(N | \text{cl}_N(K)) - \varepsilon(M_0 | \text{cl}_{M_0}(K)), \\ \sigma_3(N) &= \rho(N) - \rho(M_0), \\ \sigma_4(N) &= \nu(N) - \nu(M_0), \text{ and} \\ \sigma(N) &= (\sigma_1, \sigma_2, \sigma_3, \sigma_4)\end{aligned}$$

Note that $\nu(N) \leq \mu(N) \leq \rho(N) \leq q^{t+c+1}$ and $\varepsilon(N | \text{cl}_N(K)) \leq q^{t+c}$, so $\sigma(N) \in \mathbf{I}^4$ for any such minor N . Let (i_1, i_2, i_3, i_4) be maximal in the lexicographic order on \mathbf{I}^4 such that there is a minor M of M_0 of rank at least $n(i_1, i_2, i_3, i_4)$ for which $r_M(K) = t$, $M/K \in \mathcal{P}_q(j, h(i_1, i_2, i_3, i_4))$, and $\sigma(M, K) \geq_{\text{lex}} (i_1, i_2, i_3, i_4)$. Subject to the choice of (i_1, i_2, i_3, i_4) , choose such an M for which $|M|$ is as large as possible. It is clear from the monotonicity of $h(\cdot)$ and $n(\cdot)$ that $\sigma(M) = (i_1, i_2, i_3, i_4)$. Let $r = r(M)$; we have $r \geq n(i_1, i_2, i_3, i_4) \geq n_0$.

We argue that M (or some single-element contraction of M) is the required matroid by proving a succession of claims that use the maximality of (i_1, i_2, i_3, i_4) to show that M is very highly symmetric and modular with respect to K . Let G be a simplification of M/K , so $r(G) = r - t + j$ and $|G| = \frac{q^{r-t+j}-1}{q-1} \geq q^{r-t+j-1}$. Let $Y = \{y \in E(G) : a_{M,K}(y) < \rho(M)\}$.

6.1.1. $|Y| < \frac{1}{2}|G|$.

Proof. Suppose otherwise, so $|Y| \geq \frac{1}{2}|G| \geq q^{-1}|G| \geq q^{r-t+j-2}$.

Let $x_0 \in E(G)$ satisfy $a_{M,K}(x_0) = \rho(M)$. For each $y \in Y$, let $\{y, y', x_0\}$ be a triangle of G . The line $\text{cl}_M(\{x, y'\})$ is skew in M to K but not to $K \cup y$, and contains at most one element of $A_{M,K}(y)$. Since $a_{M,K}(x_0) > a_{M,K}(y)$, there is thus some $x_y \in A_{M,K}(x_0)$ such that $\text{cl}_M(\{x_y, y'\})$ does not intersect $A_{M,K}(y)$, from which it follows that $y' \in A_{M/x_y, K}(y)$ and so $a_{M/x_y, K}(y) > a_{M,K}(y)$. Since $a_{M,K}(x_0) \leq q^{t+c+1}$, there is some $v \in A_{M,K}(x_0)$ and a set $Y' \subseteq Y$ such that $|Y'| \geq q^{-t-c-1}|Y|$ and $x_y = v$ for all $y \in Y'$. Note that $v \notin Y'$. Let G' be a simplification of G/v ; note that $G' \in \mathcal{P}_q(k, h(i_1, i_2, i_3, i_4) - 1)$. For each $x \in E(G')$, let $L_x = \text{cl}_G(\{x, v\}) - \{v\}$. We have $a_{M/v, K}(x) = a_{M,K}(f)$ for all $f \in L_x$, so

$$a_{M/v, K}(x) = q^{-1} \sum_{f \in L_x} a_{M/f, K}(f)$$

$$\geq q^{-1} \left(\sum_{f \in L_x} a_{M,K}(f) + |Y' \cap L_x| \right).$$

Using the fact that the sets $\{L_x : x \in E(G')\}$ partition $E(G \setminus v)$, we can sum over all $x \in E(G')$ to obtain

$$\begin{aligned} \sum_{x \in E(G')} a_{M/v,K}(x) &\geq q^{-1} \left(\sum_{f \in E(G \setminus v)} a_{M,K}(f) + |Y'| \right) \\ &= q^{-1}(|G|\mu(M) - a_{M,K}(v)) + q^{-1}|Y'| \\ &\geq |G'|\mu(M) - q^{c+t} + q^{-c-t-2}|Y| \\ &\geq |G'|\mu(M) - q^{r-2t-c+j-5} + q^{-c-t-2}q^{r-t+j-2} \\ &= |G'|\mu(M) + (q-1)\delta q^{r-t-1+j} \\ &> (\mu(M, K) + \delta)|G'|, \end{aligned}$$

where we use $|G'| = \frac{q^{r-t+j-1}-1}{q-1} < q^{-1}|G|$ and $\delta = q^{-c-t-4}$, as well as our lower bound for $|Y|$, and the fact that $r \geq n_0$ so $c+t \leq r-2t-c+j-5$.

This gives $\sigma_1(M/v) > i_1$, and so $\sigma(M/v) \geq_{\text{lex}} (i_1 + 1, 0, 0, 0) >_{\text{lex}} (i_1, i_2, i_3, i_4)$. Now we have $r_{M/v}(K) = t$ and, since $h(i_1, i_2, i_3, i_4) - 1 \geq h(i_1 + 1, 0, 0, 0)$, we also have $(M/v)/K \in \mathcal{P}_q(j, h(i_1 + 1, 0, 0, 0))$. Moreover $r(M/v) = r(M) - 1 \geq n(i_1, i_2, i_3, i_4) - 1 \geq n(i_1 + 1, 0, 0, 0)$ so M/v gives a contradiction to the maximality of (i_1, i_2, i_3, i_4) . \square

6.1.2. $\nu(M) = \mu(M) = \rho(M)$.

Proof of claim: Suppose otherwise, so $\nu(M) \leq \rho(M) - 1$. Let $X = E(G) - Y$. We have $|X| \geq \frac{1}{2}|G|$; since

$$r(G) \geq n(i_1, i_2, i_3, i_4) - t \geq f_{3.6}(q, j, h(i_1, i_2, i_3, i_4), n(i_1, i_2, i_3, i_4 + 1)),$$

Lemma 3.6 implies that there is an independent set C of G so that $G/C \in \mathcal{P}_q(j, h(i_1, i_2, i_3, i_4))$, $r(G/C) \geq n(i_1, i_2, i_3, i_4 + 1)$, and each parallel class of G/C contains an element of X . It follows that $r_{M/C}(K) = t$ and $M/(C \cup K) \in \mathcal{P}_q(j, h(i_1, i_2, i_3, i_4 + 1))$ (using monotonicity of $h(\cdot)$), and every parallel class of G/C contains some $f \in X$ for which $a_{M/C,K}(f) \geq a_{M,K}(f) = \rho(M)$. So $\nu(M/C) \geq \rho(M) > \nu(M)$. This implies that $\sigma_1(M/C) \geq \sigma_1(M)$, $\sigma_3(M/C) \geq \sigma_3(M)$ and $\sigma_4(M/C) > \sigma_4(M)$. Clearly $\sigma_2(M/C) \geq \sigma_2(M)$; therefore $\sigma(M/C) \geq_{\text{lex}} (i_1, i_2, i_3, i_4 + 1)$, so M/C gives a contradiction to the maximality of (i_1, i_2, i_3, i_4) . \square

6.1.3. For every line L of M and every $f \in E(M) - \text{cl}_M(K)$, the pairs $(\text{cl}_M(K), L)$ and $(\text{cl}_M(K \cup \{f\}), L)$ are modular in M .

Proof. If one of these pairs is not modular, then there is a line L of M such that either

- (a) $L \cap \text{cl}_M(K) = \emptyset$ and $\Pi_M(L, K) = 1$, or
- (b) $\Pi_M(L, K) = 0$ and $L \cap \text{cl}_M(K \cup \{f\}) = \emptyset$ and $\Pi_M(L, K \cup \{v\}) = 1$ for some $v \in E(M) - \text{cl}_M(K \cup \{v\})$.

In either case, there is some $u \in L - \text{cl}_M(K)$. Now $r_{M/u}(K) = t$ and $(M/u)/K \in \mathcal{P}_q(j, h(i_1, i_2, i_3, i_4) - 1)$. Since $a_{M/u, K}(f) \geq a_{M, K}(f) = \mu(M)$ for each $f \in E(G) - \{u\}$, we have

$$\rho(M/u) \geq \mu(M/u) \geq \mu(M) = \rho(M),$$

which implies that $\sigma_1(M/u) \geq \sigma_1(M)$ and $\sigma_3(M/u) \geq \sigma_3(M)$. It is clear that $\sigma_2(M/u) \geq \sigma_2(M)$. Moreover, we have $\varepsilon(M/u | \text{cl}_{M/u}(K)) > \varepsilon(M | \text{cl}_M(K))$ in case (a) and $\rho(M/u) \geq a_{M/u, K}(v) > a_{M, K}(v) = \rho(M)$ in case (b), so either $\sigma_2(M) > i_2$ or $\sigma_3(M) > i_3$. In either case, $\sigma(N) \geq_{\text{lex}} (i_1, i_2, i_3 + 1, 0)$. As in the proof of the first claim, we use $h(i_1, i_2, i_3 + 1, 0) > h(i_1, i_2, i_3, i_4)$ and $n(i_1, i_2, i_3, i_4) > n(i_1, i_2, i_3 + 1, 0)$ to obtain a contradiction to the maximality of (i_1, i_2, i_3, i_4) . \square

Recall that $u = 10(c + 2) + t$. By the definitions of $h(\cdot)$ and $n(\cdot)$, we have $r(M) \geq 2u + t$ and $M/K \in \mathcal{P}_q(j, u)$. It now follows from 6.1.3 and Lemma 3.8 that $M \setminus D \in \mathcal{P}_q(k)$ for some k and some $D \subseteq \text{cl}_M(K)$. Note that $k < c$. Choose D so that $|D|$ is as small as possible; it follows from this choice that D is a union of parallel classes of $M | \text{cl}_M(K)$. Let d be an integer so that $\varepsilon(M \setminus D) = \frac{q^{r+k-1}}{q-1} - qd$, where $0 \leq d \leq \frac{q^{2k-1}}{q^2-1} < q^{2c-1}$.

6.1.4. $\mu(M) = q^{t+k-j}$ and $\varepsilon(M | (\text{cl}_M(K) - D)) = \frac{q^{t+k+j-1}}{q-1} - qd$.

Proof. We have $\varepsilon(M) = \frac{q^{r+k-1}}{q-1} - qd + \varepsilon(M | D)$. Let $d' = \varepsilon(M | D)$ and $d'' = \varepsilon(M | (\text{cl}_M(K) - D))$. Now

$$\begin{aligned} \frac{q^{r+k-1}}{q-1} - qd &= \varepsilon(M \setminus D) \\ &= \varepsilon(M/K) \mu(M) + \varepsilon(M | (\text{cl}_M(K) - D)) \\ &= \frac{q^{r-t+j-1}}{q-1} \mu(M) + d''. \end{aligned}$$

Rearranging gives

$$\begin{aligned} \mu(M) &= \frac{q^{r-k-1}}{q^{r-t+j-1}} - \frac{(q-1)(d''+d)}{q^{r-t+j-1}} \\ &= q^{k+t-j} - \frac{q^{k+t-j-1} - (q-1)(d''+d)}{q^{r-t+j-1}}. \end{aligned}$$

Since $|d'' + d| \leq q^{c+t+1} + q^{2c}$ and $r \geq n_0$, the second ‘error’ term has absolute value less than $\frac{1}{2}$; since $\mu(M) = \rho(M) \in \mathbb{Z}$, it follows that $\mu(M) = q^{k+t-j}$. Therefore the error term is exactly zero, giving $(q-1)(d'' + d) = q^{k+t+j} - 1$. The claim now follows. \square

Since $i_1 = \sigma_1(M)$, we have $\mu(M_0) \leq \mu(M) - \delta i_1$, so

$$\begin{aligned}
\varepsilon(M_0) &= \mu(M_0)\varepsilon(M_0/K) + \varepsilon(M_0|\text{cl}_{M_0}(K)) \\
&\leq (q^{k+t-j} - \delta i_1) \frac{q^{r_0-t+j-1}}{q-1} + d' + d'' - i_2 \\
&= \frac{q^{r_0+k-1}}{q-1} - \frac{q^{k+t-j-1}}{q-1} - \delta i_1 \frac{q^{r_0+t-j-1}}{q-1} + d' + d'' - i_2 \\
&= \frac{q^{r_0+k-1}}{q-1} - qd + d'' - \delta i_1 \frac{q^{r_0+t-j-1}}{q-1} - i_2. \\
&= \frac{q^{r_0+k-1}}{q-1} - qd + d'' - i_1 q^{t+c+1} - i_2,
\end{aligned}$$

where we use $r_0 \geq n_0$. Since $(i_1, i_2, i_3, i_4) \geq_{\text{lex}} (0, 0, 0, 0)$, we have $i_1 \geq 0$, and either $i_2 \geq 0$ or $i_1 > 0$. Now $i_2 \leq q^{t+c+1}$, so in either case we have $q^{c+t+1}i_1 + i_2 \geq 0$, with equality only if $i_1 = i_2 = 0$. Thus

$$\varepsilon(M_0) \leq \frac{q^{r_0+k-1}}{q-1} - qd + d'',$$

with equality only if $i_1 = i_2 = 0$ and $\mu(M_0) = \mu(M)$. We now claim that either D is empty, or another matroid M' satisfies the lemma's conclusion:

6.1.5. *Either $D = \emptyset$, or M_0 has a minor M' of rank at least m such that $M' \in \mathcal{P}_q(k+1)$ and $\frac{q^{r(M')+k+1-1}}{q-1} - \varepsilon(M') < \frac{q^{r(M_0)+k+1-1}}{q-1} - \varepsilon(M_0)$.*

Proof of claim: Suppose that $D \neq \emptyset$. Note that D contains no loops by its minimality; let $x \in D$ and $M' = M/x \setminus (D-x)$. Since $M \setminus D \in \mathcal{P}_q(k)$ and $M \setminus (D-x) \notin \mathcal{P}_q(k)$, we have $M' \in \mathcal{P}_q(k+1)$, and $k+1 < c$, so $\varepsilon(M') \geq \frac{q^{(r-1)+k+1-1}}{q-1} - q^{2c}$. Now

$$\begin{aligned}
\varepsilon(M_0) - \varepsilon(M') &\leq \frac{q^{r_0+k-1}}{q-1} + q^{t+c} - \left(\frac{q^{r+k-1}}{q-1} - q^{2c} \right) \\
&= \frac{q^{k+1}}{q-1} (q^{r_0} - q^{r-1}) + q^{2c} + q^{c+t} - q^{r_0+k} \\
&< \frac{q^{k+1}}{q-1} (q^{r_0} - q^{r(M')}),
\end{aligned}$$

since $q^{r_0+k} \geq q^{n_0+k} > q^{2c} + q^{c+t}$. Now, since $M' \in \mathcal{P}_q(k+1)$ and $r(M') = r-1 \geq n(i_1, i_2, i_3, i_4) - 1 \geq m$, by rearranging the calculation above we see that M' satisfies the claim. \square

We may thus assume that $D = \emptyset$ and so $d'' = 0$ and $M \in \mathcal{P}_q(k)$. This gives $\varepsilon(M) = \frac{q^{r+k-1}}{q-1} - qd$ and $\varepsilon(M_0) \leq \frac{q^{r_0+k-1}}{q-1} - qd$ with equality only if $i_1 = i_2 = 0$ and $\mu(M) = \mu(M_0)$. If equality does not hold, then $\frac{q^{r+k-1}}{q-1} - \varepsilon(M) < \frac{q^{r_0+k-1}}{q-1} - \varepsilon(M_0)$, so M satisfies the lemma. Assume that equality holds; since $(0, 0, 0, 0) \leq_{\text{lex}} (i_1, i_2, i_3, i_4) = (0, 0, i_3, i_4)$ and $\mu(M) = \rho(M)$, we have

$$0 \leq i_3 = \rho(M) - \rho(M_0) \leq \mu(M) - \mu(M_0) = 0,$$

so $\rho(M_0) = \mu(M_0) = \mu(M)$. Therefore $\rho(M_0) = \nu(M_0) = \mu(M)$, so $\sigma(M) = (i_1, i_2, i_3, i_4) = (0, 0, 0, 0)$, and now $M = M_0$ by the maximality of $|M|$. \square

7. THE MAIN RESULT

For each prime power q , let

$$D_q = \{(k, d) \in \mathbb{Z}_{\geq 0}^2 : 0 \leq d \leq q^{\frac{2k-1}{q^2-1}}\}.$$

For each $(k, d) \in D_q$, let $\mathcal{P}_q^d(k)$ be the set of matroids M in $\mathcal{P}_q(k)$ satisfying $\varepsilon(M) = \frac{q^{r(M)+k-1}}{q-1} - d$. (Note that $\mathcal{P}_q^d(k) = \emptyset$ unless d is a multiple of q .) Define an ordering \prec on D_q by $(k, d) \prec (k', d')$ if and only if $(k, -d) <_{\text{lex}} (k', -d')$. If $(k, d) \prec (k', d')$, then the matroids in $\mathcal{P}_q^{d'}(k')$ are ‘denser’ than those in $\mathcal{P}_q^d(k)$.

We now prove a technical lemma that combines Lemmas 5.1 and 6.1, and will easily imply our main theorem.

Lemma 7.1. *There is a computable function $f_{7.1}: \mathbb{Z}^3 \rightarrow \mathbb{Z}$ so that, for each prime power q and all integers $c, m \geq 0$, if $\mathcal{M} \in \mathbf{E}_q$ satisfies $c_{\mathcal{M}} \leq c$, and $M \in \mathcal{M}$ is such that $r(M) \geq f_{7.1}(q, c, m)$ and $\varepsilon(M) \geq \frac{q^{r(M)+k-1}}{q-1} - d$ for some $(k, d) \in D_q$, then either*

- $M \in \mathcal{P}_q(k)$, or
- there exists $(k', d') \in D_q$ with $(k', d') \succ (k, d)$ and a minor M' of M with $r(M') \geq m$ and $M' \in \mathcal{P}_q^{d'}(k')$.

Proof. Let q be a prime power, and $c, m \geq 0$. Define a sequence h_0, \dots, h_c recursively by

$$h_\ell = \max \left\{ f_{6.1}(q, c, t, j, m) : 0 \leq j \leq c, \quad 0 \leq t \leq m + c + \sum_{i=0}^{\ell-1} h_i \right\}$$

for each $\ell \in \{0, \dots, c\}$. (The summation is zero for $\ell = 0$.) Let $n_2 = \max_{0 \leq i \leq c} h_i$. Let $n_1 = \max(c, f_{4.6}(q, c, q^{-1}, n_2))$ and let $n_0 = \max(m, f_{4.2}(q, c, c+1, m))$. Set $f_{7.1}(q, c, m) = n_0$.

Let $\mathcal{M} \in \mathbf{E}_q$ be such that $c_{\mathcal{M}} \leq c$. Write $g(n)$ for $\frac{q^{n+k-1}}{q-1} - d$; note that, since $k < c$ and $(k, d) \in D_q$, we have $g(n) > qg(n-1) \geq q^{n-1}$ for all $n \geq c$. Let $M_0 \in \mathcal{M}$ satisfy $r(M_0) \geq n_0$ and $\varepsilon(M_0) \geq g(r(M_0))$. By Lemma 4.2, we see that M_0 has a weakly round restriction M_1 such that $r(M_1) \geq n_1$ and either $M_1 = M_0$ or $\varepsilon(M_1) \geq g(r(M_1)) + 1$.

By Theorem 4.6, M_1 has an $\text{AG}(n_2 - 1, q)$ -restriction R . Let M_2 be a minimal contraction-minor of M_1 such that

- R is a restriction of M_2 , and
- either $M_2 = M_0$ or $\varepsilon(M_2) \geq g(r(M_2)) + 1$.

Note that $r(M_1) \geq r(R) > c$, so $g(r(M_1)) > qg(r(M_1) - 1)$. By minimality of M_2 , each $e \in E(M_1) - \text{cl}_M(E(R))$ thus satisfies

$$\varepsilon(M_2/e) \leq g(r(M_2/e)) < q^{-1}g(r(M_2)) \leq q^{-1}\varepsilon(M_2).$$

Since M_1 is weakly round, so is M_2 ; Lemma 5.1 implies that there is a set $C_1 \subseteq E(M_2)$ so that $r_{M_2}(C_1) \leq t_1$ and M_2/C_1 has a spanning projective geometry restriction G . Note that $r(G) \geq n_2 - t_1 \geq m + 2c$; let $M'_2 = M_2/C_1$. Let C_2 be a maximal flat of M'_2 disjoint from $E(G)$ and let $s = r_{M'_2}(C_2)$. The maximality of C_2 implies that $M'_2/C_2 \in \mathcal{P}_q(s)$, so we have $s < c$.

By maximality of C_2 , every element of M'_2 is spanned by $C_2 \cup \{e\}$ for some $e \in E(G)$, and so $(M'_2/C_2)|E(G) \in \mathcal{P}_q(j)$, as C_2 spans no element of G . Let j be a minimal nonnegative integer so that there exists $C_3 \subseteq E(G)$ for which $r_{M'_2}(C_3) \leq \sum_{i=0}^{s-j-1} h_i$ and $r_{M'_2/C_3}(C_2) = j$. Clearly $0 \leq j \leq s$. If $j > 0$, then the minimality of j implies that every flat of G/C_3 of rank at most h_{s-j} is skew to C_2 in M'_2/C_3 ; since $h_j \geq 1$, this implies that $M'_2/(C_2 \cup C_3) \in \mathcal{P}_q(s, h_{s-j})$. If $j = 0$ then $M'_2/(C_2 \cup C_3) \in \mathcal{P}_q(0) = \mathcal{P}_q(0, h_s)$, so the same conclusion holds. Let $K = C_1 \cup C_2 \cup C_3$ and $t = r_{M_2}(K)$. We have

$$t \leq r_{M_2}(C_1) + r_{M'_2}(C_2) + r_{M'_2}(C_3) \leq m + c + \sum_{i=0}^{s-j-1} h_i.$$

Therefore K is a rank- t set in M_2 such that $M_2/K \in \mathcal{P}_q(s, h_{s-j})$, where $f_{6.1}(q, c, t, j, m) \leq h_{s-j}$ by the definition of h_{s-j} . Now $r(M_2) \geq r(G) = m_1 \geq h_{s-j} \geq f_{6.1}(q, c, t, j, m)$; it follows from Lemma 6.1 that M_2 has a minor $M_3 \in \mathcal{P}_q(k')$ for some $k' \in \mathbb{Z}$, such that $r(M_3) \geq m$ and either $M_3 = M_2$ or

$$\frac{q^{r(M_3)+k'}-1}{q-1} - \varepsilon(M_3) < \frac{q^{r(M_2)+k'}-1}{q-1} - \varepsilon(M_2).$$

Let $\varepsilon(M_3) = \frac{q^{r(M_3)+k'}-1}{q-1} - d'$. If $k' < k$ then the left hand side above is $d' \geq 0$ and the right hand side is at most $\frac{q^{r(M_2)+k'}-q^{r(M_2)+k}}{q-1} + d \leq d - q^{r(M_2)+k-1} < q^{2k} - q^{c+k} \leq 0$, a contradiction. If $k' > k$, then $(k', d') \succ (k, d)$, so $M' = M_3$ satisfies the lemma; assume that $k = k'$.

If $M_3 \neq M_2$, then the above inequality gives $d' < d$, so $(k', -d') \succ (k, -d)$; again, $M' = M_3$ will do. If $M_3 = M_2$, then we either have $M_3 = M_0$ (in which case $M_0 \in \mathcal{P}_q(k)$ and the first outcome holds) or

$$\frac{q^{r(M_3)+k}-1}{q-1} - d' = \varepsilon(M_3) \geq g(r(M_3)) + 1 = \frac{q^{r(M_3)+k}-1}{q-1} - (d-1).$$

Therefore $0 \leq d' < d$, so $(k, d') \in D_q$ and $(k, d') \succ (k, d)$, so $M' = M_3$ satisfies the lemma. \square

We now use Lemma 7.1 to prove a slightly stronger version of our main result, Theorem 1.2. To see that the statement below implies Theorem 1.2, observe that every $\mathcal{M} \in \mathbf{E}_q$ contains the class $\mathcal{P}_q^0(0)$ of all $\text{GF}(q)$ -representable matroids, but is disjoint from $\mathcal{P}_q(k)$ for all $k \geq c_{\mathcal{M}}$. It follows easily from maximality that the integers c, k, d_0, m all exist for \mathcal{M} . The advantage of the version stated below is that it gives a computable bound on when the ‘sufficiently large’ condition in Theorem 1.2 comes into effect, provided $q, c_{\mathcal{M}}$ and m are known; this will be useful in the next section.

Theorem 7.2. *There is a computable function $f_{7.2}: \mathbb{Z}^3 \rightarrow \mathbb{Z}$ so that, for every prime power q and all integers $c, m \geq 0$, if $\mathcal{M} \in \mathbf{E}_q$ satisfies $c_{\mathcal{M}} \leq c$, and $(k, d_0) \in D_q$ is such that $\mathcal{M} \cap \mathcal{P}_q^{d_0}(k)$ contains matroids of arbitrarily large rank, but $\mathcal{M} \cap \mathcal{P}_q^{d'}(k')$ contains no matroid of rank at least m for any $D_q \ni (k', d') \succ (k, d_0)$, then*

- $d_0 = qd$ is a multiple of q ,
- $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq f_{7.1}(q, c, m)$, and
- for all $M \in \mathcal{M}$ such that $\varepsilon(M) = \frac{q^{r(M)+k}-1}{q-1} - qd$ and $r(M) \geq f_{7.2}(q, c, m)$, we have $M \in \mathcal{P}_q^{qd}(k)$.

Proof. For each prime power q and all integers $c, m \geq 0$, set $f_{7.2}(q, c, m) = n_0 = \max(q^{58c^4}, f_{7.1}(q, c, m))$.

Let $\mathcal{M} \in \mathbf{E}_q$ be such that $c_{\mathcal{M}} \leq c$ and let $m \geq 0$ and $(k, d_0) \in D_q$ satisfy the conditions in the hypothesis for M . Note that $k \leq c$. By Lemma 3.4 and the fact that $\mathcal{M} \cap \mathcal{P}_q^{d_0}(k)$ contains matroids of arbitrarily large rank, we have $d_0 = qd$ for some d , and $h_{\mathcal{M}}(n) \geq \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq q^{58k^4}$, and therefore for all $n \geq n_0$. If $r(M) \geq n_0$ and $\varepsilon(M) \geq \frac{q^{r(M)+k}-1}{q-1} - qd$, then Lemma 7.2 implies that either $M \in \mathcal{P}_q^{qd}(k)$, or there is some $(k', d') \succ (k, qd)$ for which M has a minor $M' \in \mathcal{P}_q^{d'}(k')$ of rank at least m . The latter outcome contradicts the choice of k, d_0, m , so $M \in \mathcal{P}_q^{qd}(k)$ for any such M . Therefore $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq n_0$, and any matroid in \mathcal{M} of rank $n \geq n_0$ whose number of points attains this function is in $\mathcal{P}_q^{qd}(k)$. This gives the theorem. \square

8. COMPUTABILITY

In this section we will prove Theorem 1.3. Our first step is a technical result that shows that, if a class satisfies three particular conditions then its growth rate function can be determined completely with a finite computation and access to a membership oracle. The third condition

is that \mathcal{M} is closed under a particular type of modular sum; essentially, given a matroid $M \in \mathcal{P}_{q,k} \cap \mathcal{M}$ and a spanning projective geometry G , we should be able to ‘extend G into larger rank’ and remain in \mathcal{M} .

Lemma 8.1. *Let q be a prime power, and let $\mathcal{M} \in \mathbf{E}_q$ be a class for which there are integers $\ell, b, s \geq 0$ such that*

- $U_{2,\ell+2} \notin \mathcal{M}$,
- $\text{PG}(s-1, q') \notin \mathcal{M}$ for all $q' \in \{q+1, \dots, \ell\}$, and
- For all $k \geq 0$, if $M \in \mathcal{M} \cap \mathcal{P}_q(k)$ and $G \cong \text{PG}(n-1, q)$ are matroids such that $r(M) \geq b$, $E(G) \cap E(M) = F$ and $G|F = M|F \cong \text{PG}(r(M)-1, q)$, then $G \oplus_m M \in \mathcal{M}$.

Then there are integers k, d, n_0 , all computable given q, ℓ, b and s and a membership oracle for \mathcal{M} , such that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq n_0$, and so that every matroid $M \in \mathcal{M}$ with $r(M) = r \geq n_0$ and $\varepsilon(M) = h_{\mathcal{M}}(r)$ satisfies $M \in \mathcal{P}_q^{qd}(k)$.

Proof. Let $c = c_{\mathcal{M}}$, noting that c is computable from q, ℓ and s by Lemma 4.1. Let $m = 58c^4$ and let $n_0 = f_{7.2}(q, c, m)$.

Let $(k, qd) \in D_q$ be maximal with respect to \prec such that \mathcal{M} contains a simple rank- m matroid in $\mathcal{P}_q^{qd}(k)$. Note that k and d can be determined with at most $2^{2^{q^{m+c}}}$ queries to a membership oracle for \mathcal{M} , since every simple rank- m matroid in $\mathcal{P}_q(k)$ has at most $q^{m+k} \leq q^{m+c}$ elements.

8.1.1. $h_{\mathcal{M}}(n) \geq \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq m$.

Proof of claim: We may assume that $k > 0$. Let M be a k -element projection of $\text{PG}(m+k-1, q)$ so that $\text{si}(M) \in \mathcal{M} \cap \mathcal{P}_q^{qd}(k)$. Let \widehat{M} be a rank- $(m+k)$ matroid and K be a k -element independent set of \widehat{M} such that $\widehat{M} \setminus K \cong \text{PG}(m+k-1, q)$ and $\widehat{M}/K = M$. Let F be a rank- m flat of \widehat{M} that is skew to K (so $M|F = \widehat{M}|F \cong \text{PG}(m-1, q)$), let $G \cong \text{PG}(n+k-1, q)$ be a matroid with $\widehat{M}|F$ as a restriction such that $E(G) \cap E(\widehat{M}) = F$, and let $N = \widehat{M} \oplus_F G$. Now $\text{si}(N/K) \cong \text{si}(M \oplus_F G) \in \mathcal{M}$, since $m \geq b$.

Let J be a maximal independent set of $\widehat{M} \setminus K$ that is skew to F in \widehat{M} , such that $M/J \in \mathcal{P}_q^{qd}(k)$. By skewness to F we have $|J| \leq k$; if $|J| < k$ then by maximality, every element of x of \widehat{M} is in $\text{cl}_{\widehat{M}}(J \cup F)$, or satisfies $M/x \notin \mathcal{P}_q^{qd}(k)$. \widehat{M} has at most $\frac{q^{m+k-1}-1}{q-1}$ elements of the first type, and at most q^{58c^4} of the second type by Lemma 3.2, but

$$|\widehat{M}| \geq \frac{q^{m+k}-1}{q-1} = \frac{q^{m+k-1}-1}{q-1} + q^{m+k-1} \geq \frac{q^{m+k-1}-1}{q-1} + q^{58c^4},$$

a contradiction. So $|J| = k$. Now $M/J \in \mathcal{P}_q^{qd}(k)$ and \widehat{M}/J is a matroid in which every element outside K is parallel to an element of F ; it follows that $(\widehat{M}/J)|(F \cup K)$ is a simplification of \widehat{M}/J , and K is a rank- k independent set of \widehat{M}/J spanning no element of F . Moreover, $N/J = G \oplus_m (M/J)$; by modularity, the set K spans no element of G in N/J , so no two elements of $G \setminus F$ are parallel in $N/(J \cup K)$. Thus

$$\begin{aligned} \varepsilon(\widehat{N}/(J \cup K)) &= |G \setminus F| + \varepsilon(\widehat{M}/(J \cup K)) \\ &= \frac{q^{n+k} - q^m}{q-1} + \left(\frac{q^{(m-k)+k-1}}{q-1} - qd \right) \\ &= \frac{q^{n+k} - 1}{q-1} - qd. \end{aligned}$$

Since $\text{si}(N/K) \in \mathcal{M}$ and $r(N/K) = n + k$, the matroid $\text{si}(N/(J \cup K))$ is a rank- n matroid in \mathcal{M} , and the claim follows. \square

Now, if $M \in \mathcal{M}$ satisfies $r(M) \geq n_0$ and $\varepsilon(M) \geq \frac{q^{r(M)+k-1}}{q-1} - qd$, then either $M \in \mathcal{P}_q^{qd}(k)$, or M has a minor of rank at least m in $\mathcal{P}_q^{d'}(k')$ for some $(k', d') \succ (k, d)$. Since $m = q^{58c^4} \geq q^{58(k')^4}$, the former possibility would imply by Lemma 3.2 that M has a rank- m minor in $\mathcal{P}_q^{d'}(k')$, contradicting the maximality of (k, d) . Therefore $M \in \mathcal{P}_q^{qd}(k)$ for all such M ; this implies the lemma. \square

The next lemma shows that, when we exclude some truncation of a projective geometry as a minor, the class of matroids without a given minor is closed under the modular sum operation of Lemma 8.1.

Lemma 8.2. *Let q be a prime power, let $n, r, t, k \geq 0$ be integers with $n \geq r \geq k(t+1) + r(N)$, and let N be a simple matroid. If M is a rank- r matroid with no $T(\text{PG}(t, q))$ -minor and no N -minor such that $M \in \mathcal{P}_q(k)$, and $G \cong \text{PG}(n-1, q)$ satisfies $G|Y = M|Y \cong \text{PG}(r-1, q)$, where $Y = E(G) \cap E(M)$, then $G \oplus_m M$ has no N -minor.*

Proof. Let K be a rank- k independent flat of a matroid \widehat{M} so that $M = \widehat{M}/K$ and $\widehat{M} \setminus K \cong \text{PG}(r+k-1, q)$. For each $x \in K$, let F_x be the unique minimal flat of $\widehat{M} \setminus K$ spanning x . We have $(\widehat{M}/x)|F_x \cong T(\text{PG}(r_{\widehat{M}}(F_x) - 1, q))$, and so $M|F_x$ has a $T(\text{PG}(r_M(F_x) - 1, q))$ -restriction, implying that $r_M(F_x) \leq t$ for each $x \in K$; therefore $r_{\widehat{M}}(F_x) \leq t + k$. If F is the flat of $\widehat{M} \setminus K$ spanned by $\cup_{x \in K} F_x$, then we thus have $r_{\widehat{M}}(F) \leq k(t+k)$ and $K \subseteq \text{cl}_{\widehat{M}}(F)$; note that $r_{\widehat{M}}(F) \leq r(\widehat{M}) - r(N)$ by hypothesis.

It follows from the construction of F that

$$\widehat{M} = (\widehat{M} \setminus K) \oplus_m (\widehat{M}|(K \cup F)).$$

Let $\widehat{G} \cong \text{PG}(n+k-1, q)$ be a matroid so that $\widehat{M} \setminus K$ is a restriction of \widehat{G} to a rank- $(r+k)$ flat, and G is a restriction of \widehat{G} to a rank- n flat that intersects $E(\widehat{M})$ in a rank- r flat that is skew to K in \widehat{M} . By choice of \widehat{G} , we have

$$G \oplus_m Y \cong ((\widehat{G} \oplus_m (\widehat{M}|(K \cup F)))/K)|(E(M) \cup E(G)).$$

It therefore suffices to show that, if $M' = \widehat{G} \oplus_m \widehat{M}|(K \cup F)$, then M'/K has no N -minor. Suppose that $N \cong (M'/C \setminus D)/K$, where C is independent and D is co-independent in M' . Now $r(N) = r(M') - |C| - r_{M'}(K)$ and so $|C| \geq n - k - r(N)$. Moreover, $|C \cap F| \leq r_{M'}(F) \leq kt$, so $|C - F| \geq n - k - r(N) - kt \geq n - r$, so C contains an $(n-r)$ -element independent set C' of M' that is skew to F in M' . However $\text{si}(M'/C') \cong \text{si}(\widehat{M})$, and $M = \widehat{M}/K$ has no N -minor, contradicting the fact that $(M'/C')/K$ has an N -minor. \square

The next result, proved in ([8], Theorem 3.4) will easily imply that matroids representable over given field are also closed under the modular sum operation.

Theorem 8.3. *Let \mathbb{F} be a field with a $\text{GF}(q)$ -subfield. If M is an \mathbb{F} -representable matroid with a $\text{PG}(n-1, q)$ -restriction for some n , then M has an \mathbb{F} -representation so that every column in this restriction has entries only in $\text{GF}(q)$.*

We now restate and prove Theorem 1.3.

Theorem 8.4. *Let \mathcal{F} be a finite set of finite fields and \mathcal{O} be a finite set of simple matroids. Let \mathcal{M} be the class of matroids in $\text{Ex}(\mathcal{O})$ that are representable over all fields in \mathcal{F} . If \mathcal{M} is base- q exponentially dense and does not contain all truncations of $\text{GF}(q)$ -representable matroids, then there are computable nonnegative integers k, d and n_0 such that $h_{\mathcal{M}}(n) = \frac{q^{n+k}-1}{q-1} - qd$ for all $n \geq n_0$.*

Proof. Since \mathcal{M} is base- q exponentially dense, there is some matroid $U_{2,\ell+2}$ that is either in \mathcal{O} , or not representable over a field in \mathcal{F} ; clearly ℓ is computable given \mathcal{F} and \mathcal{O} . Now q is the largest prime power so that $\text{GF}(q)$ is a subfield of all fields in \mathcal{F} and \mathcal{O} contains no $\text{GF}(q)$ -representable matroid. If $\mathcal{F} \neq \emptyset$, then for each $\mathbb{F} \in \mathcal{F}$, the matroid $T(\text{PG}(|\mathbb{F}|, |\mathbb{F}|))$ has a $U_{|\mathbb{F}|, |\mathbb{F}|+2}$ -restriction so is not in \mathcal{M} . If $\mathcal{F} = \emptyset$ then \mathcal{O} must contain some minor of a truncation of a projective geometry over $\text{GF}(q)$ and must therefore contain a spanning restriction of such a truncation. Therefore $T(\text{PG}(t, q)) \notin \mathcal{M}$, where t is either the minimum size of a field in \mathcal{F} , or the maximum rank of a matroid in \mathcal{O} if \mathcal{F} is empty. Finally, since \mathcal{M} is base- q exponentially dense, it is easy to

check that we have $\text{PG}(s-1, q') \notin \mathcal{M}$ for every prime power $q' \in \{q+1, \dots, \ell\}$, where s is the maximum rank of a matroid in \mathcal{O} .

By Lemma 4.1, we can compute the constant $c_{\mathcal{M}}$. To show that n_0 , k and d are computable for \mathcal{M} , it suffices to show that there exists $b \geq 0$ so that \mathcal{M} is closed under the modular sum operation in Lemma 8.1. Since each $\mathbb{F} \in \mathcal{F}$ has a $\text{GF}(q)$ -subfield, it is clear by Lemma 8.3 that we can adjoin some \mathbb{F} -representation of a matroid $M \in \mathcal{M}$ to a $\text{GF}(q)$ -representation of a projective geometry G to obtain an \mathbb{F} -representation of their modular sum, so the sum operation preserves \mathbb{F} -representability. Since $k \leq c_{\mathcal{M}}$, the fact that \mathcal{M} is itself closed under the sum operation for $b = \max\{r(N) + c_{\mathcal{M}}(t+1) : N \in \mathcal{O}\}$, follows from Lemma 8.2. Since it is easy to decide membership of a given matroid in \mathcal{M} , Lemma 8.1 now gives the theorem. \square

Excluding a truncation as a minor is fundamental to the proof of Lemma 8.2 and therefore to Theorem 1.3. Conjecture 1.4 claims that without this exclusion, it is impossible to compute the growth rate function in general for a class defined by excluded minors. Our motivation for this conjecture is the drastic difference between the complexity of describing a k -element projection with and without excluding some truncation as a minor, which we now outline.

The proof of Lemma 8.2 uses the fact that, for $n \geq k(t+k)$, every k -element extension of $\text{PG}(n+k-1, q)$ with no $T(\text{PG}(t, q))$ -minor is in fact the modular sum of $\text{PG}(n+k-1, q)$ and some k -element extension of $\text{PG}(k(t+k)-1, q)$. Each such extension has at most $q^{k(t+k)}$ elements, and thus the number of nonisomorphic k -element extensions of $\text{PG}(n+k-1, q)$ with no $T(\text{PG}(t, q))$ -minor (and therefore the number of rank- n matroids in $\mathcal{P}_q(k)$ with no $T(\text{PG}(t, q))$ -minor) is at most $2^{2q^{k(t+k)}}$, a bound independent of n .

On the other hand, let $G \cong \text{PG}(n+1, q)$, let \mathcal{F} be the set of rank- n flats of G , and let $\mathcal{F}' \subseteq \mathcal{F}$. It is routine to show that there is a unique two-element extension $M = M_{\mathcal{F}'}$ of G by elements x_1 and x_2 such that

- $M \setminus x_i$ is a free extension of G for each $i \in \{1, 2\}$,
- $\{x_1, x_2\}$ is skew to every flat of rank less than n in G , and is spanned by no hyperplane of G , and
- the set of rank- n flats of G skew to $\{x_1, x_2\}$ is exactly \mathcal{F}' .

Since different sizes of \mathcal{F}' correspond to nonisomorphic matroids, this gives at least $|\mathcal{F}| = \frac{(q^{n+2}-1)(q^{n+1}-1)}{(q^2-1)(q-1)} > q^{2n}$ nonisomorphic two-element extensions of G . (Actually the number of nonisomorphic M is the number of orbits of the action of $\text{Aut}(G)$ on $2^{\mathcal{F}}$, which seems likely to

be doubly exponential in n .) These extensions will correspond to non-isomorphic rank- n matroids in $\mathcal{P}_q(2)$, and their abundance markedly contrasts the constant number we get when excluding a truncation.

The complexity of even these two-element extensions leads us to believe that one can perhaps encode undecidable problems in the form of minor-testing on two-element projections of projective geometries; this motivates Conjecture 1.4.

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